

MIXED VOLUME PRESERVING FLOW BY POWERS OF HOMOGENEOUS CURVATURE FUNCTIONS OF DEGREE ONE

SHUNZI GUO

ABSTRACT. This paper concerns the evolution of a closed hypersurface of dimension $n(\geq 2)$ in the Euclidean space \mathbb{R}^{n+1} under a mixed volume preserving flow. The speed equals a power $\beta(\geq 1)$ of homogeneous, either convex or concave, curvature functions of degree one plus a mixed volume preserving term, including the case of powers of the mean curvature and of the Gauss curvature. The main result is that if the initial hypersurface satisfies a suitable pinching condition, there exists a unique, smooth solution of the flow for all times, and the evolving hypersurfaces converge exponentially to a round sphere, enclosing the same mixed volume as the initial hypersurface. This result covers and generalises the previous results for convex hypersurfaces in the Euclidean space by McCoy [34] and Cabezas-Rivas and Sinestrari [12] to more general curvature flows for convex hypersurfaces with similar curvature pinching condition.

1. Introduction

Let M^n be a smooth, compact oriented manifold of dimension $n(\geq 2)$ without boundary, and $X_0 : M^n \rightarrow \mathbb{R}^{n+1}$ a smooth immersion of M^n into the Euclidean space. We are interested in a one-parameter family of smooth immersions: $X_t : M^n \rightarrow \mathbb{R}^{n+1}$ evolving according to

$$(1.1) \quad \begin{cases} \frac{\partial}{\partial t} X(p, t) = \{\bar{\phi}(t) - \Phi(F(\mathcal{W}(p, t)))\} \nu(p, t), & p \in M^n, \\ X(\cdot, 0) = X_0(\cdot), \end{cases}$$

where $\nu(p, t)$ is the outer unit normal to $M_t = X_t(M^n)$ at the point $X(p, t)$, $\mathcal{W}_\nu(p, t) = -\mathcal{W}'_\nu(p, t)$ is the matrix of the Weingarten map on the tangent space TM^n induced by X_t , Φ is a smooth supplementary function and the $F(\mathcal{W})$ will be specified below. The $\bar{\phi}(t)$ is a mixed volume preserving global term, and stands for the averaged Φ in the sense of the mixed volumes of a convex hypersurface M_t :

$$(1.2) \quad \bar{\phi}_m(t) = \frac{\int_{M_t} E_{m+1} \Phi(F) d\mu_t}{\int_{M_t} E_{m+1} d\mu_t},$$

for each $m = -1, 0, 1, \dots, n-1$. Here $d\mu_t$ denotes the surface area element of M_t , and E_m is the m th elementary symmetric functions,

$$(1.3) \quad E_m = \begin{cases} 1 & m = 0 \\ \sum_{1 \leq i_1 < \dots < i_m \leq n} \lambda_{i_1} \cdots \lambda_{i_m} & m = 1, \dots, n. \end{cases}$$

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Obviously $E_1 = H$ and $E_n = K$, where H and K denote the mean curvature and the Gauß-Kronecker curvature respectively. For each fixed m , as is clear from the presence of the global term $\bar{\phi}(t)$ in equation (1.1), the flow keeps the mixed volumes of Ω_t , where Ω_t is the solid $(n+1)$ -region bounded by M_t .

$$(1.4) \quad V_{n-m}(\Omega_t) = \begin{cases} \text{Vol}(\Omega_t) & m = -1 \\ \left\{ (n+1) \binom{n}{m} \right\}^{-1} \int_{M_t} E_m d\mu_t & m = 0, 1, \dots, n-1 \end{cases}$$

constant. Here $\text{Vol}(\Omega_t)$ denotes the volume of the domain enclosed by M_t .

We require that the $F(\mathcal{W})$ satisfies the following properties:

- Conditions 1.1.*
- (i). $F = f \circ \lambda$, where λ is the map which takes a self-adjoint operator to its ordered eigenvalues, and f is a smooth, symmetric function defined on an open symmetric cone $\Gamma \subseteq \Gamma_+$ containing $\{(c, \dots, c) : c > 0\}$, where $\Gamma_+ = \{\lambda : \lambda_i > 0, i = 1, \dots, n\}$.
 - (ii). f is strictly monotone: $\frac{\partial f}{\partial \lambda_i} > 0$ for each $i = 1, \dots, n$, at each point of Γ .
 - (iii). f is homogeneous of degree one: $f(k\lambda) = kf(\lambda)$ for any $k > 0$.
 - (iv). f is strictly positive on Γ and normalised to have $f(1, \dots, 1) = 1$.
 - (v). Either:
 - f is convex, or
 - f is concave and one of the following hold
 - (a) f approaches zero on the boundary of Γ ,
 - (b) f is inverse concave, that is, the function $\tilde{f}(x_1, \dots, x_n) = -f(x_1^{-1}, \dots, x_n^{-1})$ is concave.

Examples of $F(\mathcal{W})$ fulfilling the conditions 1.1 include, apart from the normalization:

- Examples 1.2.*
- The concave and inverse concave examples of curvature functions $F: \left(\frac{E_m}{E_l}\right)^{\frac{1}{m-l}}$, $n \geq m > l \geq 0$, or the power means $(\sum_{i=1}^n \lambda_i^r)^{\frac{1}{r}}$ for $|r| \leq 1$. For a proof of the inverse concavity of these functions see the proofs of [4, Theorem 2.6, Theorem 2.7].
 - The convex examples of curvature functions F , for a proof that these curvature functions are convex, see [35, p. 105].:
 - (1) the mean curvature $H = \sum_{i=1}^n \lambda_i$,
 - (2) the length of the second fundamental form $|A| = \sqrt{\sum_{i=1}^n \lambda_i^2}$ and the completely symmetric functions $\gamma_k = \left(\sum_{|\beta|=k} \lambda^\beta\right)^{\frac{1}{k}}$, $1 \leq k \leq n$, where β is a multiindex, $\beta \in \mathbb{N}^n$, and $\lambda^\beta = \lambda_1^{\beta_1} \cdot \lambda_2^{\beta_2} \dots \lambda_n^{\beta_n}$.

There exists a wide literature about curvature problems of the form (1.1), for which the speed $\Phi(F) = F$ satisfies Conditions 1.1 for hypersurfaces in different ambient spaces. First, in the case of the volume-preserving mean curvature flow, Huisken [26] showed that convex Euclidean hypersurfaces remain convex for all time and converge exponentially fast to round spheres (the corresponding result for curves in the Euclidean plane is due to Gage [20]), while Andrews [3] extended this result to the smooth anisotropic mean curvature flow, and McCoy showed similar results for the Euclidean surface area preserving mean curvature flow [32] and the mixed volume preserving mean curvature flows [33]. The volume-preserving flow has

been used to study constant mean curvature surfaces between parallel planes [6, 7] and to find canonical foliations near infinity in asymptotically flat spaces arising in general relativity [27] (Rigger [36] showed analogous results in the asymptotically hyperbolic setting). If the initial hypersurface is sufficiently close to a fixed Euclidean sphere (possibly non-convex), Escher and Simonett [16] proved that the flow converges exponentially fast to a round sphere, a similar result for average mean convex hypersurfaces with initially small traceless second fundamental form is due to Li [29]. For a general ambient manifold, Alikakos and Freire [1] proved long time existence and convergence to a constant mean curvature surface under the hypotheses that the initial hypersurface is close to a small geodesic sphere and that it satisfies some non-degenerate conditions. Cabezas-Rivas and Miquel exported the Euclidean results of [6, 7] to revolution hypersurfaces in a rotationally symmetric space [11], and showed the same results as Huisken [26] for a hyperbolic background space [10] by assuming the initial hypersurface is horospherically convex. On the other hand, there are few results on speeds different from the mean curvature: McCoy [34] proved the convergence to a sphere for a large class of function F homogeneous of degree one (including the case $\Phi(F) = E_m^\beta$ with $m\beta = 1$), Makowski showed the volume preserving curvature flow in Lorentzian manifolds for F as a function with homogeneous of degree one exponential converges to a hypersurface of constant F -curvature [31] (moreover, stability properties and foliations of such a hypersurface was also examined). In 2010, Cabezas-Rivas and Sinestrari [12] studied the deformation of hypersurfaces with initial pinching condition (1.7) in \mathbb{R}^{n+1} by a speed with the form $\Phi(F) = E_m^\beta$ for some power $\beta \geq 1/m$. In this way $\Phi(F)$ is a homogeneous function of the curvatures with a degree $m\beta \geq 1$. The results of [12] do not closely relate to the ambient space. Recently, The author, together with Li and Wu [19], achieved such extension of the above Theorem 1.3 of Cabezas-Rivas and Sinestrari [12] to convex hypersurfaces with similar pinching condition in the hyperbolic case.

Inspired by these results of [12, 18, 34], in this paper we replace the speed function $F(\lambda)$ in [34] by a power of the $F(\lambda)$, namely we study the long time existence and the convergence of the flow (1.1) with the speed Φ given by a power of the curvature function, namely

$$(1.5) \quad \Phi(F) = F^\beta$$

for some $\beta \geq 1$. In this way Φ is a homogeneous function of the curvatures with a degree $\beta \geq 1$. We call the flow (1.1) with this choice of speed *the mixed volume preserving F^β -flow*. Our analysis will focused on the existence of this deformation of convex hypersurfaces and its limiting behavior. It is well-known that preserving convexity of hypersurfaces is a key point to analyze the convergence of the flow. However, we face the first difficulty: Due to higher homogeneity of the speed, we cannot directly apply the maximum principle of Hamilton [24] to the evolution of the principal curvature to show that convexity of hypersurfaces is preserved. Thus, we have to search some “nice” geometric quantities of the principal curvature for which the maximum principle can be applied to the evolution. So the next problem we meet is: what is the suitable geometric quantity for the mixed volume preserving F^β -flow. Meanwhile we observe that most of the literature in the investigation of evolution equations, requires an additional assumption of a suitable geometric quantity on the initial hypersurface for example, of K/H^n used for the gauss curvature flow [13], the power mean curvature flow [37, 18] and the volume preserving

E_m^β -flow [12, 19], R/H^2 used for the flow by the square root of the scalar curvature [14], and K/F^n used for the mixed volume preserving F -flow [34]. We realized that the two scalars K/F^n for convex F and K/H^n for concave F should be good candidates for our flow. In Section 4, we shall succeed in estimating the two scalar. In this paper, by imposing additional assumptions on the two geometric quantities mentioned above on the initial hypersurface, we shall prove

Theorem 1.3. *Suppose F satisfies Conditions 1.1, and for $\beta \geq 1$ there exists a positive constant $C = C(n, \beta)$ such that the following holds: If the initial hypersurface of \mathbb{R}^{n+1} is pinched at every point in the case that*

(i). *for convex F ,*

$$(1.6) \quad K(p) > CF^n > 0,$$

(ii). *for concave F ,*

$$(1.7) \quad K(p) > CH^n > 0,$$

then the flow (1.1)-(1.2) with Φ given by (1.5) has a unique and smooth solution for all times, inequality (1.6) or (1.7) remains true everywhere on the evolving hypersurfaces M_t for all $t > 0$ and the M_t 's converge, as $t \rightarrow \infty$, exponentially in the C^∞ -topology, to a round sphere enclosing the same value of V_{n-m} as M_0 .

Remark 1.4. Since the main result of Theorem 1.3 cover the previously known results in \mathbb{R}^{n+1} : of the volume-preserving mean curvature flow [26], the surface area preserving mean curvature flow [32], the mixed volume preserving mean curvature flows [33] and the volume-preserving E_m^β -flow [12]. Nevertheless, our results are a significant extension of those results in this direction.

Our work is mainly motivated by the approaches in the papers [12, 34], we make modifications to consider our problem for the different speed (1.5). The rest of the paper is organized as follows: Section 2 first gives some useful preliminary results employed in the remainder of the paper. Section 3 contains details of the short time existence of the flow (1.1)-(1.2) and the induced evolution equations of some important geometric quantities. In Section 4 applying the maximum principle to the evolution equation of the two quantities K/F^n for convex F and K/H^n for concave F gives that if the initial hypersurface is pinched good enough then this is preserved for $t > 0$ as long as the flow (1.1)-(1.2) exists. This is a fundamental step in our procedure as in most of the literature quoted above. Furthermore, Section 5 proves the uniform bound of the speed by following a method which was firstly used by Tso [42]. Using more sophisticated results for fully nonlinear elliptic and parabolic partial differential equations, Section 6 obtains uniform bounds on all derivatives of the curvature and proves long time existence of the flow (1.1)-(1.2). Finally Section 7, following the ideas in [40] and [5], obtains the lower speed bounds, which will then allow us to prove that these evolving hypersurfaces converge to a sphere of \mathbb{R}^{n+1} smoothly and exponentially.

2. Notations and preliminary results

From now on, we will follow similar notation used in [12]. In particular, in local coordinates $\{x^i\}$, $1 \leq i \leq n$, near $p \in M^n$ and $\{y^\alpha\}$, $0 \leq \alpha, \beta \leq n$, near $F(p) \in \mathbb{R}^{n+1}$. Denote by a bar all quantities on \mathbb{R}^{n+1} , for example by $\bar{g} = \{\bar{g}_{\alpha\beta}\}$

the metric, by $\bar{g}^{-1} = \{\bar{g}^{\alpha\beta}\}$ the inverse of the metric, by $\bar{\nabla}$ the covariant derivative, by $\bar{\Delta}$ the rough Laplacian. Components are sometimes taken with respect to the tangent vector fields $\partial_\alpha (= \frac{\partial}{\partial y^\alpha})$ associated with a local coordinate $\{y^\alpha\}$ and sometimes with respect to a moving orthonormal frame e_α , where $\bar{g}(e_\alpha, e_\beta) = \delta_{\alpha\beta}$. The corresponding geometric quantities on M^n will be denoted by g (the induced metric), g^{-1} , ∇ , Δ , ∂_i and e_i , etc.. Then further important quantities are the second fundamental form $A(p) = \{h_{ij}\}$ and the Weingarten map $\mathcal{W} = \{g^{ik}h_{kj}\} = \{h_j^i\}$ as a symmetric operator and a self-adjoint operator respectively. The eigenvalues $\lambda_1(p) \leq \dots \leq \lambda_n(p)$ of \mathcal{W} are called the principal curvatures of $X(M^n)$ at $X(p)$. The mean curvature is given by

$$H := \text{tr}_g \mathcal{W} = h_i^i = \sum_{i=1}^n \lambda_i,$$

the squared norm of the second fundamental form by

$$|A|^2 := \text{tr}_g(\mathcal{W}^t \mathcal{W}) = h_j^i h_i^j = h^{ij} h_{ij} = \sum_{i=1}^n \lambda_i^2,$$

and Gauß-Kronecker curvature by

$$K := \det(\mathcal{W}) = \det\{h_j^i\} = \frac{\det\{h_{ij}\}}{\det\{g_{ij}\}} = \prod_{i=1}^n \lambda_i.$$

More generally, the m th elementary symmetric functions E_m are given by

$$E_m(\lambda) = \sum_{1 \leq i_1 < \dots < i_r \leq n} \lambda_{i_1} \cdots \lambda_{i_r}, \quad \text{for } \lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n,$$

we also set $E_m = 1$ if $m = 0$. Thus the m th mean curvatures E_m are given by (1.5). In addition we define

$$\Gamma_m := \{\lambda \in \mathbb{R}^n | E_1(\lambda) > 0, E_2(\lambda) > 0, \dots, E_m(\lambda) > 0\}$$

and the quotients

$$Q_m = \frac{E_m(\lambda)}{E_{m-1}(\lambda)}, \quad \forall \lambda \in \Gamma_m.$$

Since E_m is homogeneous of degree m , the speed F is homogeneous of degree β in the curvatures λ_i . Denote the vector $(\lambda_1, \dots, \lambda_n)$ of \mathbb{R}^n by λ and the positive cone by $\Gamma_+ \subset \mathbb{R}^n$, i.e.

$$\Gamma_+ = \{\lambda = (\lambda_1, \dots, \lambda_n) : \lambda_i > 0, \forall i\}.$$

It is clear that H, K, E_m, Q_m, F may be viewed as functions of λ , or as functions of A , or as functions of \mathcal{W} , or also functions of space and time on M_t . Throughout this paper we sum over repeated indices from 1 to n unless otherwise indicated. In computations on the hypersurface M_t , raised indices indicate contraction with the metric.

We will denote by \dot{F}^{kl} the matrix of the first partial derivatives of F with respect to the components of its argument:

$$\left. \frac{\partial}{\partial s} F(A + sB) \right|_{s=0} = \dot{F}^{ij}(A) B_{ij}.$$

Furthermore we denote the second partial derivatives of F by

$$\left. \frac{\partial^2}{\partial s^2} F(A + sB) \right|_{s=0} = \ddot{F}^{kl,rs}(A) B_{kl} B_{rs}.$$

If we do not indicate explicitly where derivatives of F and of f are evaluated then they are evaluated at \mathcal{W} and $\lambda(\mathcal{W})$ respectively. We will further use the shortened notation $\dot{f}^i = \frac{\partial f}{\partial \lambda_i}$ and $\ddot{f}^{ij} = \frac{\partial^2 f}{\partial \lambda_i \partial \lambda_j}$ where appropriate. We will use similar notation and conventions for other functions of matrices when we differentiate them.

Now we state the following well-known inequalities for general curvature functions F :

Lemma 2.1. *For any concave (convex) F fulfilling the conditions 1.1,*

$$F(\lambda) \leq (\geq) \frac{F(1, \dots, 1)}{n} H(\lambda).$$

and

$$\sum_{i=1}^n f_i(\lambda) = \text{tr}(F^{kl}) \geq (\leq) 1,$$

where $\lambda = (\lambda_k) \in \Gamma_+$.

Proof. See [21, Lemma 2.2.19, Lemma 2.2.20]. Also see [43] for concave F . \square

We need the following relation between the first derivatives of F and the first derivatives of f , of [34, Corollary 3.2].

Lemma 2.2. *If \mathcal{W} has distinct eigenvalues λ_k . then F is concave (convex) at \mathcal{W} if and only if f is concave (convex) at $\lambda(\mathcal{W})$ and for all $k \neq l$*

$$\frac{\dot{f}_k - \dot{f}_l}{\lambda_k - \lambda_l} \leq (\geq) 0.$$

We also need the following inequalities for F . See also [34, Corollary 3.3] for a derivation.

Lemma 2.3. *For any concave (convex) F at \mathcal{W} fulfilling the conditions 1.1,*

$$H \text{tr}_F(A\mathcal{W}) - F|A|^2 \leq 0 (\geq 0).$$

In our analysis of the flow equations we will use several geometric estimates for a compact, strictly convex hypersurface with suitably pinched curvatures, which implies suitably pinching relation between outer radius (circumradius) and inner radius (inradius) for enclosed region by the hypersurface. The first of these appears in (see also [34]).

Lemma 2.4. *If a smooth, compact, uniformly convex manifold M^n satisfies everywhere the pointwise curvature pinching estimate*

$$\lambda_n \leq C_1 \lambda_1$$

for some constant $C_1 < \infty$, then the circumradius ρ_+ of M^n satisfies

$$\rho_+ \leq C_2 \rho_-$$

where $C_2 = \left(\frac{n+2}{\sqrt{2}}\right)C_1$ and ρ_- denotes the inradius of M^n .

The ratio of circumradius to inradius of M^n gives estimates for the inradius and circumradius in terms of the mixed volumes of M^n .

Lemma 2.5. *Under the assumptions of Lemma 2.4, for any $m = 1, \dots, n+1$,*

$$\rho_- \geq \frac{1}{C_2} \left(\frac{V_m}{\omega_{n+1}} \right)^{\frac{1}{m}} \quad \text{and} \quad \rho_+ \leq C_2 \left(\frac{V_m}{\omega_{n+1}} \right)^{\frac{1}{m}}.$$

where ω_{n+1} denotes the volume of the $(n+1)$ -dimensional unit ball.

Proof. We can argue exactly as in [34, Corollary 3.6]. \square

We note some important properties of E_m (see [12] for a simple derivation).

Lemma 2.6. *Let $1 \leq m \leq n$ be fixed.*

- i) *The m th roots $E_m^{1/m}$ are concave in Γ_+ .*
- ii) *For all i , $\frac{\partial E_m}{\partial \lambda_i}(\lambda) > 0$, where $\lambda \in \Gamma_+$.*
- iii) *$E_m^{1/m} \leq \frac{H}{n}$; equivalently, $E_m \leq \left(\frac{H}{n} \right)^{m\beta}$.*

The following algebraic property proved by Schulze in [38, Lemma 2.5], will be needed in the later sections.

Lemma 2.7. *For any $\varepsilon > 0$ assume that $\lambda_i \geq \varepsilon H > 0$, $i = 1, \dots, n$, at some point of an n -dimensional hypersurface. Then at the same point there exists a $\delta = \delta(\varepsilon, n) > 0$ such that*

$$\frac{n|A|^2 - H^2}{H^2} \geq \delta \left(\frac{1}{n^n} - \frac{K}{H^n} \right).$$

In our analysis we need some a priori estimates on the Hölder norms of the solutions to elliptic and parabolic partial differential equations in Euclidean spaces. We recall that, in the case of a function depending on space and time, there is a suitable definition of Hölder norm which is adapted to the purposes of parabolic equations (see e.g. [30]). In addition to the standard Schauder estimates for linear equations, we use in the paper some more recent results which are collected here. The estimates below hold for suitable classes of weak solutions; for the sake of simplicity, we state them in the case of a smooth classical solution, which is enough for our purposes.

Given $r > 0$, we denote by B_r the ball of radius $r > 0$ in \mathbb{R}^n centered at the origin. First we recall a well known result due to Krylov and Safonov, which applies to linear parabolic equations of the form

$$(2.1) \quad \left(a^{ij}(x, t) D_i D_j + b^i(x, t) D_i + c(x, t) - \frac{\partial}{\partial t} \right) u = f$$

in $B_r \times [0, T]$, for some $T > 0$. We assume that $a^{ij} = a^{ji}$ and that a^{ij} is uniformly elliptic; that is, there exist two constants $\lambda, \Lambda > 0$ such that

$$(2.2) \quad \lambda |v|^2 \leq a^{ij}(x, t) v_i v_j \leq \Lambda |v|^2$$

for all $v \in \mathbb{R}^n$ and all $(x, t) \in B_r \times [0, T]$. Then the following estimate holds [28, Theorem 4.3]:

Theorem 2.8. *Let $u \in C^2(B_r \times [0, T])$ be a solution of (2.1), where the coefficients are measurable, satisfy (2.2) and*

$$|b^i|, |c| \leq \lambda_1 \quad \text{for all } i = 1, \dots, n,$$

for some $\lambda_1 > 0$. Then, for any $0 < r' < r$ and any $0 < \delta < T$ we have

$$\|u\|_{C^\alpha(B_{r'} \times [\delta, T])} \leq C (\|u\|_{C(B_r \times [0, T])} + \|f\|_{L^\infty(B_r \times [0, T])})$$

for some constants $C > 0$ and $\alpha \in (0, 1)$ depending on $n, \lambda, \Lambda, \lambda_1, r, r'$ and δ .

Next we quote a result for fully nonlinear elliptic equations, which is due to Caffarelli. We consider the equation

$$(2.3) \quad F(D^2u(x), x) = f(x), \quad x \in B_r.$$

Here $F : \mathcal{S} \times B_r \rightarrow \mathbb{R}$, where \mathcal{S} is the set of the symmetric $n \times n$ matrices. The nonlinear operator F is called uniformly elliptic if there exist $\Lambda \geq \lambda > 0$ such that

$$(2.4) \quad \lambda \|B\| \leq F(A + B, x) - F(A, x) \leq \Lambda \|B\|$$

for any $x \in B_r$ and any pair $A, B \in \mathcal{S}$ such that B is nonnegative definite.

Theorem 2.9. *Let $u \in C^2(B_r)$ be a solution of (2.3), where F is continuous and satisfies (2.4). Suppose in addition that F is concave with respect to D^2u for any $x \in B_r$. Then there exists $\bar{\alpha} \in (0, 1)$ with the following property: if, for some $\lambda_2 > 0$ and $\alpha \in (0, \bar{\alpha})$, we have that $f \in C^\alpha(\Omega)$ and that*

$$F(A, x) - F(A, y) \leq \lambda_2 |x - y|^\alpha (\|A\| + 1), \quad x, y \in B_r, \quad A \in \mathcal{S},$$

then, for any $0 < r' < r$, we have the estimate

$$\|u\|_{C^{2+\alpha}(B_{r'})} \leq C (\|u\|_{C(B_r)} + \|f\|_{C^\alpha(B_r)} + 1)$$

where $C > 0$ only depends on $n, \lambda, \Lambda, \lambda_2, r$ and r' .

The above result follows from Theorem 3 in [8] (see also Theorem 8.1 in [9] and the remarks thereafter). It generalizes, by a perturbation method, a previous estimate, due to Evans and Krylov, about equations with concave dependence on the hessian. In contrast with Evans-Krylov result (see e.g. inequality (17.42) in [22]), Theorem 2.8 gives an estimate in terms of the C^α -norm of f rather than the C^2 -norm, and this is essential for our purposes.

Finally, we recall an interior Hölder estimate, due to Di Benedetto and Friedman [15, Theorem 1.3], for solutions of the degenerate parabolic equation

$$(2.5) \quad \frac{\partial v}{\partial t} - D_i (a^{ij}(x, t, Dv) D_j v^d) = f(x, t, v, Dv),$$

being $d > 1$.

Theorem 2.10. *Let $v \in C^2(B_r \times [0, T])$ be a nonnegative solution of (2.5), where a^{ij} satisfies (2.2). Let $c_1, c_2, N > 0$ be such that*

$$|f(x, t, v, Dv)| \leq c_1 |Dv^d| + c_2,$$

and

$$\sup_{0 < t < T} \|v(\cdot, t)\|_{L^2(B_r)}^2 + \|Dv^d\|_{L^2(B_r \times [0, T])}^2 \leq N.$$

Then for any $0 < \delta < T$ and $0 < r' < r$, we have

$$\|v\|_{C^\alpha(B_{r'} \times [\delta, T])} \leq C,$$

for suitable $C > 0, \alpha \in (0, 1)$ depending only on $n, N, \lambda, \Lambda, \delta, c_1, c_2, r$ and r' .

3. Short time existence and evolution equations

This section first considers short time existence for the initial value problem (1.1).

Theorem 3.1. *Let $X_0 : M^n \rightarrow \mathbb{R}^{n+1}$ be a smooth closed hypersurface with $F > 0$ everywhere. Then there exists a unique smooth solution X_t of problem (1.1)-(1.2) with Φ given by (1.5), defined on some time interval $[0, T)$, with $T > 0$.*

Proof. We can argue exactly as in [12, Theorem 3.1]; although the assumptions on the initial hypersurface and the speeds in that paper are different, the proof applies to our case as well. \square

Proceeding now exactly as in [25, 21] from the basic equation (1.1) with Φ given by (1.5) we can easily compute how the following geometric quantities on M_t evolve.

Proposition 3.2. *On any solution M_t of (1.1)-(1.2) with Φ given by (1.5) the following hold:*

$$\begin{aligned}
 (3.1) \quad & \partial_t g = 2(\bar{\phi} - \Phi)A, \\
 & \partial_t g^{-1} = -2(\bar{\phi} - \Phi)g^{-1}\mathcal{W}, \\
 & \partial_t \nu = X_*(\nabla \Phi), \\
 & \partial_t (d\mu_t) = (\bar{\phi} - \Phi)H d\mu_t, \\
 & \partial_t A = \Delta_{\dot{\Phi}} A + \ddot{\Phi}(\nabla \cdot \mathcal{W}, \nabla \cdot \mathcal{W}) + \text{tr}_{\dot{\Phi}}(A\mathcal{W})A + [\bar{\phi} - (\beta + 1)\Phi]A\mathcal{W}, \\
 (3.2) \quad & \partial_t \mathcal{W} = \Delta_{\dot{\Phi}} \mathcal{W} + \ddot{\Phi}(\nabla \cdot \mathcal{W}, \nabla \cdot \mathcal{W}) + \text{tr}_{\dot{\Phi}}(A\mathcal{W})\mathcal{W} - [\bar{\phi} + (\beta - 1)\Phi]\mathcal{W}^2.
 \end{aligned}$$

In addition, the position vector field $\mathfrak{X}(\cdot, t) = X(\cdot, t) - x_0$ (with origin x_0) on M_t evolves according to

$$(3.3) \quad \partial_t \langle \mathfrak{X}, \nu \rangle = \Delta_{\dot{\Phi}} \langle \mathfrak{X}, \nu \rangle + \text{tr}_{\dot{\Phi}}(\mathcal{W}A) \langle \mathfrak{X}, \nu \rangle + (\bar{\phi} - (\beta + 1)\Phi).$$

We are going to prove the claimed the mixed volume preservation property of the flow (1.1) with $\bar{\phi}_m(t)$, $m = -1, \dots, n-1$.

Lemma 3.3. *Any flow of the form (1.1) with $\bar{\phi} = \bar{\phi}_m(t)$ given by (1.2) preserves the mixed volume V_{n-m} .*

Proof. By the definition (1.4) of the mixed volume, we now recall from [34, Lemma 4.3], along any flow of the form (1.1),

$$\frac{d}{dt} \int_{M_t} E_m d\mu_t = \begin{cases} 0 & m = n \\ (m+1) \int_{M_t} (\bar{\phi} - \Phi) E_{m+1} d\mu_t & m = 0, 1, \dots, n-1 \end{cases}$$

Hence in view of that $\bar{\phi} = \bar{\phi}_m(t)$ given by (1.2), we get that for $m = 0, 1, \dots, n-1$

$$\frac{d}{dt} V_{n-m} = (m+1) \left\{ (n+1) \binom{n}{m} \right\}^{-1} \int_{M_t} (\bar{\phi} - \Phi) E_{m+1} d\mu_t = 0.$$

\square

In the next theorem, we derive the evolution of any homogeneous function of the Weingarten map \mathcal{W} defined on an evolving hypersurface M_t of \mathbb{R}^{n+1} under the flow (1.1).

Theorem 3.4. *If G is a homogeneous function of the Weingarten map \mathcal{W} of degree α , then the evolution equation of G under the flow (1.1) in \mathbb{R}^{n+1} is the following*

$$\begin{aligned} \partial_t G = & \Delta_{\dot{\Phi}} G - \dot{\Phi} \ddot{G}(\nabla \cdot \mathcal{W}, \nabla \cdot \mathcal{W}) + \dot{G} \ddot{\Phi}(\nabla \cdot \mathcal{W}, \nabla \cdot \mathcal{W}) + \alpha \text{tr}_{\dot{\Phi}}(A\mathcal{W}) G \\ & - [\bar{\phi} + (\beta - 1)\Phi] \dot{G} \mathcal{W}^2. \end{aligned}$$

Proof. The definition of \dot{G} and \ddot{G} allow us to write $\text{Hess}_{\nabla} G$ as follows:

$$\text{Hess}_{\nabla} G = \dot{G} \text{Hess}_{\nabla} \mathcal{W} + \ddot{G}(\nabla \cdot \mathcal{W}, \nabla \cdot \mathcal{W}),$$

which gives

$$\Delta_{\dot{\Phi}} G = \dot{G} g^{-1} \text{Hess}_{\nabla} G = \dot{G} \Delta_{\dot{\Phi}} \mathcal{W} + \dot{\Phi} \ddot{G}(\nabla \cdot \mathcal{W}, \nabla \cdot \mathcal{W}).$$

Therefore, by (3.2)

$$\begin{aligned} \partial_t G = & \dot{G} \partial_t \mathcal{W} \\ = & \dot{G} \Delta_{\dot{\Phi}} \mathcal{W} + \dot{G} \ddot{\Phi}(\nabla \cdot \mathcal{W}, \nabla \cdot \mathcal{W}) + \text{tr}_{\dot{\Phi}}(A\mathcal{W}) \dot{G} \mathcal{W} - [\bar{\phi} + (\beta - 1)\Phi] \dot{G} \mathcal{W}^2 \\ = & \Delta_{\dot{\Phi}} G - \dot{\Phi} \ddot{G}(\nabla \cdot \mathcal{W}, \nabla \cdot \mathcal{W}) + \dot{G} \ddot{\Phi}(\nabla \cdot \mathcal{W}, \nabla \cdot \mathcal{W}) \\ & + \alpha \text{tr}_{\dot{\Phi}}(A\mathcal{W}) G - [\bar{\phi} + (\beta - 1)\Phi] \dot{G} \mathcal{W}^2, \end{aligned}$$

where Euler's relation $\dot{G} \mathcal{W} = \alpha G$ is used in the last line. \square

An immediate application of the theorem above is to obtain the evolution equations for H , and F .

Proposition 3.5. *On any solution M_t of (1.1)-(1.2) with Φ given by (1.5) the following hold:*

$$\begin{aligned} \partial_t H = & \Delta_{\dot{\Phi}} H + \text{tr}[\ddot{\Phi}(\nabla \cdot \mathcal{W}, \nabla \cdot \mathcal{W})] + \text{tr}_{\dot{\Phi}}(A\mathcal{W}) H - (\bar{\phi} + (\beta - 1)\Phi) |A|^2, \end{aligned} \quad (3.4)$$

$$\begin{aligned} \partial_t H^n = & \Delta_{\dot{\Phi}} H^n - n(n-1) H^{n-2} |\nabla H|_{\dot{\Phi}}^2 + n H^{n-1} \text{tr}[\ddot{\Phi}(\nabla \cdot \mathcal{W}, \nabla \cdot \mathcal{W})] \\ & + n(\Phi - \bar{\phi}) H^{n-1} |A|^2 \end{aligned} \quad (3.5)$$

$$\begin{aligned} \partial_t F = & \Delta_{\dot{\Phi}} F - \dot{\Phi} \ddot{F}(\nabla \cdot \mathcal{W}, \nabla \cdot \mathcal{W}) + \dot{F} \ddot{\Phi}(\nabla \cdot \mathcal{W}, \nabla \cdot \mathcal{W}) \\ & + \text{tr}_{\dot{\Phi}}(A\mathcal{W}) F - (\bar{\phi} + (\beta - 1)\Phi) \text{tr}_{\dot{\Phi}}(A\mathcal{W}), \end{aligned} \quad (3.6)$$

$$\begin{aligned} \partial_t F^n = & \Delta_{\dot{\Phi}} F^n - n(n-1) F^{n-2} |\nabla F|_{\dot{\Phi}}^2 - n F^{n-1} \dot{\Phi} \ddot{F}(\nabla \cdot \mathcal{W}, \nabla \cdot \mathcal{W}) \\ & + n F^{n-1} \text{tr}_{\dot{F}}[\ddot{\Phi}(\nabla \cdot \mathcal{W}, \nabla \cdot \mathcal{W})] + n F^n \text{tr}_{\dot{\Phi}}(A\mathcal{W}) - (\bar{\phi} + (\beta - 1)\Phi) \text{tr}_{\dot{\Phi}}(A\mathcal{W}), \end{aligned} \quad (3.7)$$

$$\begin{aligned} \partial_t K = & \Delta_{\dot{\Phi}} K - \dot{\Phi} \ddot{K}(\nabla \cdot \mathcal{W}, \nabla \cdot \mathcal{W}) + \dot{K} \ddot{\Phi}(\nabla \cdot \mathcal{W}, \nabla \cdot \mathcal{W}) \\ & + [(1 - \beta)\Phi - \bar{\phi}] \dot{K} \mathcal{W}^2 + \text{tr}_{\dot{\Phi}}(A\mathcal{W}) \dot{K} \mathcal{W}, \end{aligned} \quad (3.8)$$

$$\partial_t \Phi = \Delta_{\dot{\Phi}} \Phi + (\Phi - \bar{\phi}) \text{tr}_{\dot{\Phi}}(A\mathcal{W}).$$

Furthermore, (3.7) can be rewritten as

Lemma 3.6. *On any solution M_t of (1.1)-(1.2) with Φ given by (1.5) the following holds:*

$$\begin{aligned}
 (3.9) \quad \partial_t K &= \Delta_{\dot{\Phi}} K - \frac{(n-1)}{n} \frac{|\nabla K|_{\dot{\Phi}}^2}{K} + \frac{K}{H^2} |H \nabla \mathcal{W} - \mathcal{W} \nabla H|_{\dot{\Phi}, b}^2 \\
 &\quad - \frac{H^{2n}}{nK} |\nabla(KH^{-n})|_{\dot{\Phi}}^2 + K \operatorname{tr}_b \left(\ddot{\Phi}(\nabla \mathcal{W}, \nabla \mathcal{W}) \right) \\
 &\quad + [(1-\beta)\Phi - \bar{\phi}] KH + n \operatorname{tr}_{\dot{\Phi}}(A\mathcal{W})K, \\
 (3.10) \quad &= \Delta_{\dot{\Phi}} K - \frac{(n-1)}{n} \frac{|\nabla K|_{\dot{\Phi}}^2}{K} + \frac{K}{F^2} |F \nabla \mathcal{W} - \mathcal{W} \nabla F|_{\dot{\Phi}, b}^2 \\
 &\quad - \frac{F^{2n}}{nK} |\nabla(KF^{-n})|_{\dot{\Phi}}^2 + K \operatorname{tr}_b \left(\ddot{\Phi}(\nabla \mathcal{W}, \nabla \mathcal{W}) \right) \\
 &\quad + [(1-\beta)\Phi - \bar{\phi}] KH + n \operatorname{tr}_{\dot{\Phi}}(A\mathcal{W})K,
 \end{aligned}$$

where $b := \mathcal{W}^{-1}$.

Proof. Note that

$$(3.11) \quad \dot{K} = Kb,$$

this implies

$$(3.12) \quad \dot{K} \mathcal{W}^2 = KH,$$

and

$$(3.13) \quad \dot{K} \ddot{\Phi}(\nabla \mathcal{W}, \nabla \mathcal{W}) = Kb \ddot{\Phi}(\nabla \mathcal{W}, \nabla \mathcal{W}) = K \operatorname{tr}_b \left(\ddot{\Phi}(\nabla \mathcal{W}, \nabla \mathcal{W}) \right).$$

A direct calculation as for example in Lemma 3.2 of [13] gives

$$(3.14) \quad -\dot{\Phi} \dot{K}(\nabla \mathcal{W}, \nabla \mathcal{W}) = -\frac{|\nabla K|_{\dot{\Phi}}^2}{K} - K \operatorname{tr}_{\dot{\Phi}}(\nabla b \nabla \mathcal{W})$$

and

$$(3.15) \quad -K \operatorname{tr}_{\dot{\Phi}}(\nabla b \nabla \mathcal{W}) = \frac{K}{H^2} |H \nabla \mathcal{W} - \mathcal{W} \nabla H|_{\dot{\Phi}, b}^2 + \frac{|\nabla K|_{\dot{\Phi}}^2}{nK} - \frac{H^{2n}}{nK} |\nabla(KH^{-n})|_{\dot{\Phi}}^2.$$

Therefore, identities (3.12), (3.13), (3.14) and (3.15) together apply to (3.7) to give (3.9).

Also,

$$(3.16) \quad -K \operatorname{tr}_{\dot{\Phi}}(\nabla b \nabla \mathcal{W}) = \frac{K}{F^2} |F \nabla \mathcal{W} - \mathcal{W} \nabla F|_{\dot{\Phi}, b}^2 + \frac{|\nabla K|_{\dot{\Phi}}^2}{nK} - \frac{F^{2n}}{nK} |\nabla(KF^{-n})|_{\dot{\Phi}}^2.$$

Similarly, applying identities (3.12), (3.13), (3.14) and (3.16) together to (3.7) gives (3.10). \square

4. Preserving pinching

To control the pinching of the principal curvatures along the flow (1.1) of Euclidean spaces, Schulze, in [38], following an idea of Tso [42], looked at a test function K/H^n , which was also considered in [12]. Furthermore two analogous quantities \tilde{K}/\tilde{H}^n and K/F^n were taken into consideration in [18, 19] and [34] respectively. In this section, we use test functions $Q_1 = K/H^n$ in the case of concave F and

$Q_2 = K/F^n$ in the case of convex F respectively. The rest of this section consists of showing the inequality $Q_i \geq C > 0$ remain under the evolution for $i = 1, 2$.

We begin by deriving the following evolution equations for the quantities Q_i for $i = 1, 2$.

Lemma 4.1. *For the ambient space $N^{n+1} = \mathbb{R}^{n+1}$, on any solution M_t of (1.1) the following holds:*

$$\begin{aligned}
 \partial_t Q_1 &= \Delta_{\dot{\Phi}} Q_1 + \frac{(n+1)}{nH^n} \langle \nabla Q_1, \nabla H^n \rangle_{\dot{\Phi}} - \frac{(n-1)}{nK} \langle \nabla Q_1, \nabla K \rangle_{\dot{\Phi}} - \frac{H^n}{nK} |\nabla Q_1|_{\dot{\Phi}}^2 \\
 (4.1) \quad &+ \frac{Q_1}{H^2} |H \nabla \mathcal{W} - \mathcal{W} \nabla H|_{\dot{\Phi}, b}^2 + Q_1 \operatorname{tr}_{b - \frac{n}{H} Id} \left(\ddot{\Phi}(\nabla \mathcal{W}, \nabla \mathcal{W}) \right) \\
 &+ [(\beta - 1)\Phi + \bar{\phi}] \frac{Q_1}{H} \left(n|A|^2 - H^2 \right).
 \end{aligned}$$

and

$$\begin{aligned}
 \partial_t Q_2 &= \Delta_{\dot{\Phi}} Q_2 + \frac{(n+1)}{nF^n} \langle \nabla Q_2, \nabla F^n \rangle_{\dot{\Phi}} - \frac{(n-1)}{nK} \langle \nabla Q_2, \nabla K \rangle_{\dot{\Phi}} - \frac{F^n}{nK} |\nabla Q_2|_{\dot{\Phi}}^2 \\
 (4.2) \quad &+ \frac{Q_2}{F^2} |F \nabla \mathcal{W} - \mathcal{W} \nabla F|_{\dot{\Phi}, b}^2 + Q_2 \operatorname{tr}_{b - \frac{n}{F} \dot{F}} \left(\ddot{\Phi}(\nabla \mathcal{W}, \nabla \mathcal{W}) \right) \\
 &+ n \frac{Q_2}{F} \ddot{F}(\nabla \mathcal{W}, \nabla \mathcal{W}) + [(\beta - 1)\Phi + \bar{\phi}] \frac{Q_2}{F} (n \operatorname{tr}_{\dot{F}}(A \mathcal{W}) - HF).
 \end{aligned}$$

Proof. We prove the evolution equation (4.2) for Q_2 ; the proof of the evolution equation (4.1) for Q_1 is similar by using (3.4) and (3.9). By (3.6) and (3.9)

$$\begin{aligned}
 \partial_t Q_2 &= \frac{1}{F^n} \partial_t K - \frac{1}{F^{2n}} \partial_t F^n \\
 &= \frac{\Delta_{\dot{\Phi}} K}{F^n} - \frac{K}{F^{2n}} \Delta_{\dot{\Phi}} F^n - \frac{(n-1)}{n} \frac{|\nabla K|_{\dot{\Phi}}^2}{K F^n} - \frac{Q_2}{n} |\nabla Q_2|_{\dot{\Phi}}^2 + n(n-1) \frac{Q_2}{F^2} |\nabla F|_{\dot{\Phi}}^2 \\
 (4.3) \quad &+ \frac{Q_2}{F^2} |F \nabla \mathcal{W} - \mathcal{W} \nabla F|_{\dot{\Phi}, b}^2 + Q_2 \operatorname{tr}_{b - \frac{n}{F} \dot{F}} \left(\ddot{\Phi}(\nabla \mathcal{W}, \nabla \mathcal{W}) \right) \\
 &+ n \frac{Q_2}{F} \ddot{F}(\nabla \mathcal{W}, \nabla \mathcal{W}) + [(\beta - 1)\Phi + \bar{\phi}] \frac{Q_2}{F} (n \operatorname{tr}_{\dot{F}}(A \mathcal{W}) - HF).
 \end{aligned}$$

Furthermore, the first derivative and second derivative term in (4.3) can be computed as follows, the equality

$$\nabla \left(\frac{K}{F^n} \right) = \frac{\nabla K}{F^n} - \frac{K}{F^{2n}} \nabla F^n$$

implies

$$(4.4) \quad \Delta_{\dot{\Phi}} \left(\frac{K}{F^n} \right) = \frac{\Delta_{\dot{\Phi}} K}{F^n} - 2 \frac{\langle \nabla F^n, \nabla \tilde{K} \rangle_{\dot{\Phi}}}{F^{2n}} + 2 \frac{K}{F^{3n}} |\nabla F^n|_{\dot{\Phi}}^2 - \frac{K}{F^{2n}} \Delta_{\dot{\Phi}} F^n,$$

$$(4.5) \quad \left\langle \nabla \left(\frac{K}{F^n} \right), \nabla F^n \right\rangle_{\dot{\Phi}} = \frac{\langle \nabla F^n, \nabla \tilde{K} \rangle_{\dot{\Phi}}}{F^n} - \frac{K}{F^{2n}} |\nabla F^n|_{\dot{\Phi}}^2,$$

and

$$(4.6) \quad \left\langle \nabla \left(\frac{K}{F^n} \right), \nabla K \right\rangle_{\dot{\Phi}} = \frac{|\nabla K|_{\dot{\Phi}}^2}{F^n} - \frac{K}{F^{2n}} \langle \nabla F^n, \nabla \tilde{K} \rangle_{\dot{\Phi}}.$$

From (4.4), (4.5) and (4.6), it follows

$$\begin{aligned}
 (4.7) \quad & \frac{\Delta_{\dot{\Phi}} K}{F^n} - \frac{K}{F^{2n}} \Delta_{\dot{\Phi}} F^n - \frac{(n-1)}{n} \frac{|\nabla K|_{\dot{\Phi}}^2}{K F^n} \\
 &= \Delta_{\dot{\Phi}} \left(\frac{K}{F^n} \right) + \frac{(n+1)}{n F^n} \left\langle \nabla \left(\frac{K}{F^n} \right), \nabla F^n \right\rangle_{\dot{\Phi}} \\
 &\quad - \frac{(n-1)}{n K} \left\langle \nabla \left(\frac{K}{F^n} \right), \nabla \tilde{K} \right\rangle_{\dot{\Phi}} - n(n-1) \frac{K}{F^{n+2}} |\nabla F|_{\dot{\Phi}}^2.
 \end{aligned}$$

Thus, applying equation (4.7) to (4.3) gives (4.2). \square

In order to apply the maximum principle to (4.1) and (4.2), and show that for $i = 1, 2$, $\min_{p \in M_t} Q_i(p, t)$ are non-decreasing in time some preliminary inequalities are needed in the sequel. The following elementary property is a consequence of ([12], Lemma 4.2) (see also [13] and [38]).

Lemma 4.2. *Given $\varepsilon \in (0, 1/n)$, for any convex (concave) F fulfilling the conditions 1.1, there exists a constant $C_1(\varepsilon, n)$ ($C_2(\varepsilon, n)$), such that, for any $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ with $0 < \lambda_1 \leq \dots \leq \lambda_n$,*

$$K(\lambda) > C_1 F^n(\lambda) \quad (K(\lambda) > C_2 H^n(\lambda)),$$

then, we have

$$\lambda_1 > \varepsilon n F(\lambda) \quad (\lambda_1 > \varepsilon H(\lambda)).$$

Proof. Given any $\varepsilon \in (0, 1/n)$, we define

$$\begin{aligned}
 \Gamma_\varepsilon &= \{\lambda = (\lambda_1, \dots, \lambda_n) : 0 \leq \lambda_1 \leq \varepsilon n F(\lambda)\}, \\
 \Xi_\varepsilon &= \{\lambda \in \Gamma_\varepsilon : |\lambda| = 1\}.
 \end{aligned}$$

We have $F(\lambda) > 0$ on any nonzero element of Γ_ε ; hence, the quotient K/F^n is defined everywhere on Ξ_ε . Let us call M_ε the maximum of K/F^n on Ξ_ε , which exists because Ξ_ε is compact. By homogeneity, the inequality $K \leq M_\varepsilon F^n$ is also satisfied by the elements of Γ_ε . Therefore, if $\lambda = (\lambda_1, \dots, \lambda_n)$ with $0 < \lambda_1 \leq \dots \leq \lambda_n$ is such that $K > M_\varepsilon F^n$, then λ does not belong to Γ_ε . The lemma follows by choosing $C = M_\varepsilon$. \square

The following estimate which is a stronger version of Lemma 2.3 (ii) in [25] can be viewed as a generalization of Cabezas-Rivas and Miquel in [12].

Lemma 4.3. *If F is convex and positive, and the inequality $\mathcal{W} > \varepsilon n F Id$ is valid with some $\varepsilon > 0$ at a point on a hypersurface immersed in \mathbb{R}^{n+1} , then $\varepsilon \leq 1/n$ and*

$$|F \nabla \mathcal{W} - \mathcal{W} \nabla F|^2 \geq \frac{n-1}{2} \varepsilon^2 F^2 |\nabla \mathcal{W}|^2.$$

Proof. Proceeding as in [12, Lemma 4.1], we first observe that the assumption implies that $\lambda_1 \geq \varepsilon n F > 0$. It follows from this, Conditions 1.1 (iii) and (iv) that

$$F(\lambda_1, \dots, \lambda_n) \geq \lambda_1 F(1, \dots, 1) \geq \lambda_1 \geq n \varepsilon F(\lambda_1, \dots, \lambda_n) > 0,$$

which implies that $\varepsilon \leq 1/n$. Furthermore applying Lemma 2.1 with the convexity of F to $\lambda_1 \geq n \varepsilon F$ implies that $\lambda_1 \geq \varepsilon H$. Thus we may deduce that

$$\begin{aligned}
 (4.8) \quad & h_i^j h_j^l \nabla^i F \nabla_l F \leq \lambda_n^2 |\nabla F|^2 \leq (|A|^2 - (n-1) \lambda_1^2) |\nabla F|^2 \\
 & \leq (|A|^2 - (n-1) \varepsilon^2 H^2) |\nabla F|^2 \leq (1 - (n-1) \varepsilon^2) |A|^2 |\nabla F|^2.
 \end{aligned}$$

Now we can write

$$(4.9) \quad |F\nabla\mathcal{W} - \mathcal{W}\nabla F|^2 = |\nabla\mathcal{W}|^2 F^2 + |\nabla F|^2 |A|^2 - 2F\langle \mathcal{W}\nabla F, \nabla\mathcal{W} \rangle$$

whose last term, in local coordinates, takes the following form

$$-2F\nabla_i h_{jl} \nabla^i F h^{jl} = -F\nabla_i h_{jl} (\nabla^i F h^{jl} + \nabla^j F h^{il}),$$

by the Codazzi equations. Using this and the inequality $\langle U, V \rangle \leq \frac{2-\varepsilon'}{2}|U|^2 + \frac{1}{2(2-\varepsilon')}|V|^2$, with $U = F\nabla_i h_{jl}$, $V = \nabla^i F h^{jl} + \nabla^j F h^{il}$ and $\varepsilon' = (n-1)\varepsilon^2$, one estimate

$$\begin{aligned} 2F\langle \mathcal{W}\nabla F, \nabla\mathcal{W} \rangle &\leq \frac{2-\varepsilon'}{2}F^2|\nabla\mathcal{W}|^2 + \frac{1}{2(2-\varepsilon')}|\nabla^i F h^{jl} + \nabla^j F h^{il}|^2 \\ &= (1 - \frac{\varepsilon'}{2})F^2|\nabla\mathcal{W}|^2 + \frac{1}{2-\varepsilon'}(|\nabla F|^2|A|^2 + \nabla_i F h_{jl} \nabla^j F h^{il}) \\ &\stackrel{(4.8)}{\leq} (1 - \frac{\varepsilon'}{2})F^2|\nabla\mathcal{W}|^2 + |\nabla F|^2|A|^2, \end{aligned}$$

and the conclusion follows from (4.9). \square

The estimate above is also valid in the case of concave F , see also [12, Lemma 4.1].

Lemma 4.4. *If F is concave and positive, and the inequality $\mathcal{W} > \varepsilon HId$ is valid with some $\varepsilon > 0$ at a point on a hypersurface immersed in \mathbb{R}^{n+1} , then $\varepsilon \leq 1/n$ and*

$$|H\nabla\mathcal{W} - \mathcal{W}\nabla H|^2 \geq \frac{n-1}{2}\varepsilon^2 H^2 |\nabla\mathcal{W}|^2.$$

Also as in [12], the preceding three lemmas allow us to prove the pinching estimate for our flow, which is one of the key steps in the proof of our main result.

Theorem 4.5. *There exists a constant $C^* = C(n, \beta)$ with the following property: if $X : M \times (0, T) \rightarrow \mathbb{R}^{n+1}$, with $t \in (0, T)$, is a smooth solution of (1.1)–(1.2) with $\Phi(F)$ given by (1.5) for some $\beta \geq 1$, where F satisfies Conditions 1.1 (i), (ii), (iii), and $f(1, \dots, 1) = 1$, such that*

- *the initial immersion X_0 satisfies (1.6) for F convex ((1.7) for F concave) with the constant C^* ,*
- *the solution $M_t = X(M, t)$ satisfies $F > 0$ ($F > 0$) for all times $t \in (0, T)$,*

then the minimum of Q_2 (Q_1) on M_t is nondecreasing in time.

Proof. We prove the convex F case; the concave F case is similar. The assumption $F > 0$ on initial hypersurface ensures that the quotient Q_2 is well-defined for $t \in (0, T)$. For proof of the theorem, it is sufficient to prove that the minimum of Q_2 (denote by \mathcal{Q}) is nondecreasing in time. First, applying (1.6) with the assumption $F > 0$ to Lemma 4.2 implies that $\lambda_1 > 0$ on M_t for $t = 0$, then this implies that $\lambda_1 > 0$ on M_t for $t \in (0, T)$. In fact, suppose to the contrary that there exists a first time $t_0 > 0$ at which $\lambda_1 = 0$ at some point, then $\mathcal{Q}(t_0) = 0$. On the other hand, since the theorem holds in the convex case, $\mathcal{Q}(t)$ is nondecreasing in $(0, t_0)$, so it cannot decrease from C^* to zero which gives a contradiction. Now applying the maximum principle to equation (4.2) for \mathcal{Q} gives

$$\begin{aligned}
 \partial_t Q_2 &\geq \frac{Q_2}{F^2} |F\nabla\mathcal{W} - \mathcal{W}\nabla F|_{\dot{\Phi},b}^2 + Q_2 \operatorname{tr}_{b-\frac{n}{F}\dot{F}} \left(\ddot{\Phi}(\nabla\mathcal{W}, \nabla\mathcal{W}) \right) \\
 &\quad + [(\beta-1)\Phi + \bar{\phi}] \frac{Q_2}{F} (n\operatorname{tr}_{\dot{F}}(A\mathcal{W}) - HF). \\
 (4.10) \quad &\geq Q_2 \left\{ \frac{1}{F^2} |F\nabla\mathcal{W} - \mathcal{W}\nabla F|_{\dot{\Phi},b}^2 - \left| b - \frac{n}{F}\dot{F} \right| \left| \ddot{\Phi}(\nabla\mathcal{W}, \nabla\mathcal{W}) \right| \right. \\
 &\quad \left. + [(\beta-1)\Phi + \bar{\phi}] \frac{Q_2}{F} (n\operatorname{tr}_{\dot{F}}(A\mathcal{W}) - HF) \right\}.
 \end{aligned}$$

The various terms appearing here can be estimated as follows, as in [12, Theorem 4.3]. Combining the convexity of M_t and Lemma 2.3, we estimate that $n\operatorname{tr}_{\dot{F}}(A\mathcal{W}) - HF \geq 0$. Thus the third term on right hand side in inequality (4.10) can be dropped with the strictly convexity on M_t and the assumption $\beta \geq 1$. It remains to estimate the first two terms on right hand side in the inequality (4.10), now proceeding exactly as in [12], [13] and [38], choose orthonormal frame which diagonalizes \mathcal{W} so that

$$\begin{aligned}
 (4.11) \quad |F\nabla\mathcal{W} - \mathcal{W}\nabla F|_{\dot{\Phi},b}^2 &= \sum_{i,m,n} \dot{\Phi}^i \frac{1}{\lambda_m} \frac{1}{\lambda_n} (F\nabla_i F_m^n - F_m^n \nabla_i F)^2 \\
 &\geq \frac{1}{H^2} \sum_{i,m,n} \dot{\Phi}^i (F\nabla_i F_m^n - F_m^n \nabla_i F)^2
 \end{aligned}$$

where $\lambda_m \leq H$ was used in the last inequality by strictly convexity of M_t , i.e., $\lambda_m > 0$ for any m . Now the property that each $\dot{\Phi}^i$ is positive in the interior of the positive cone can be used. More precisely, for any $\varepsilon \in (0, 1/n]$, we set

$$\Xi_\varepsilon := \{ \lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n : \lambda_1 \geq \varepsilon n F > 0 \},$$

and

$$W_1(\varepsilon) = \min \{ \dot{\Phi}^i(\lambda) : 1 \leq i \leq n, \lambda \in \Xi_\varepsilon, |\lambda| = 1 \}.$$

By homogeneity of $\dot{\Phi}^i$ with degree $\beta - 1$ and Conditions 1.1 (iii), exactly as in the formula at the top of p.453 of [12], the following inequality holds:

$$\dot{\Phi}^i(\lambda) \geq W_1(\varepsilon) |\lambda|^{\beta-1}, \quad \lambda \in \Xi_\varepsilon,$$

where $W_1(\varepsilon)$ is an increasing positive function of ε . This estimation, convexity of a hypersurface and Lemma 4.3 together imply that the inequality (4.11) can be estimated as follows:

$$(4.12) \quad |F\nabla\mathcal{W} - \mathcal{W}\nabla F|_{\dot{\Phi},b}^2 \geq \frac{n-1}{2} W_1(\varepsilon) \varepsilon^2 F^2 |\mathcal{W}|^{\beta-3} |\nabla\mathcal{W}|^2,$$

for some $\varepsilon \in (0, 1/n)$.

The term $\left| \ddot{\Phi}(\nabla\mathcal{W}, \nabla\mathcal{W}) \right|$ is smooth as long as $\lambda_i > 0$ for any i , homogeneous of degree $\beta - 2$ in λ_i and quadratic in $\nabla\mathcal{W}$. Thus the following estimation of the term $\left| \ddot{\Phi}(\nabla\mathcal{W}, \nabla\mathcal{W}) \right|$ can be derived as in [12, inequality (4.7)]: For any $\varepsilon \in (0, 1/n)$, there exists a constant $W_2(\varepsilon)$ such that, at any point where $\mathcal{W} \geq \varepsilon n F Id$,

$$(4.13) \quad \left| \ddot{\Phi}(\nabla\mathcal{W}, \nabla\mathcal{W}) \right| \leq W_2(\varepsilon) |\mathcal{W}|^{\beta-2} |\nabla\mathcal{W}|^2,$$

where $W_2(\varepsilon)$ is decreasing in ε .

A next step is to show that $\left|b - \frac{n}{F}\dot{F}\right|$ is small if the principal curvatures are pinched enough. Since Lemma 2.2 with F convex implies that the inequalities $\dot{f}_1 \leq \dots \leq \dot{f}_n$. Furthermore, by Conditions 1.1 (iii), it is clear that

$$\left|b - \frac{n}{F}\dot{F}\right| \leq \sqrt{n} \max \left\{ \left(\frac{1}{\lambda_1} - \frac{n\dot{f}_1}{f} \right), \left(\frac{n\dot{f}_n}{f} - \frac{1}{\lambda_n} \right) \right\}.$$

Since for some $\varepsilon \in (0, 1/n)$

$$(4.14) \quad \lambda_1 \geq \varepsilon n f,$$

then

$$(4.15) \quad \frac{1}{\lambda_1} - \frac{n\dot{f}_1}{f} \leq \frac{1 - \varepsilon n^2 \dot{f}_1}{\varepsilon n f}.$$

Furthermore, by homogeneity of \dot{f}_1 with degree 0 and Conditions 1.1 (iii), there exists a constant $W_3(\varepsilon)$ such that, at any point where (4.14) holds,

$$\dot{f}_1(\lambda) \geq W_3(\varepsilon), \quad \lambda \in \Xi_\varepsilon,$$

where $W_3(\varepsilon)$ is an increasing positive function of ε . Thus, (4.15) can be estimated as follows:

$$(4.16) \quad \frac{1}{\lambda_1} - \frac{n\dot{f}_1}{f} \leq \frac{1 - \varepsilon n^2 W_3(\varepsilon)}{\varepsilon n f}.$$

On other hand, recalling the derivation of (4.8), the convexity of M_t implies that

$$\lambda_n \leq (1 - (n-1)\varepsilon) H$$

Thus Lemma 2.1 with F convex gives as follows:

$$\lambda_n \leq (1 - (n-1)\varepsilon) n f$$

which implies that

$$\frac{n\dot{f}_n}{f} - \frac{1}{\lambda_n} \leq \frac{(n-1)(1-n\varepsilon)}{n(1-(n-1)\varepsilon)f},$$

where we use the inequality $\dot{f}_n \leq 1$ by Lemma 2.1 with F convex. This combines with estimate (4.16) to give

$$(4.17) \quad \left|b - \frac{n}{F}\dot{F}\right| \leq \frac{\mathcal{N}(\varepsilon)}{F},$$

where

$$(4.18) \quad \mathcal{N}(\varepsilon) := \frac{1}{\sqrt{n}} \max \left\{ \frac{(1 - \varepsilon n^2 W_3(\varepsilon))}{\varepsilon}, \frac{(n-1)(1-n\varepsilon)}{(1-(n-1)\varepsilon)} \right\}.$$

Thus, recalling the facts that the convexity of F implies the inequalities $F \geq \frac{H}{n}$, and $|H| \geq |\mathcal{W}|$, estimations (4.11), (4.12), (4.13) and (4.17) together give:

$$\begin{aligned} & \frac{1}{F^2} |F \nabla \mathcal{W} - \mathcal{W} \nabla F|_{\Phi, b}^2 - \left|b - \frac{n}{F}\dot{F}\right| \left| \ddot{\Phi}(\nabla \mathcal{W}, \nabla \mathcal{W}) \right| \\ & \geq |\mathcal{W}|^{\beta-3} |\nabla \mathcal{W}|^2 \left(\frac{(n-1)}{2} W_1(\varepsilon) \varepsilon^2 - n W_2(\varepsilon) \mathcal{N}(\varepsilon) \right). \end{aligned}$$

To achieve our purpose by application of the maximum principle, it is necessary that $\mathcal{F}(\varepsilon) := |\mathcal{W}|^{\beta-3} |\nabla \mathcal{W}|^2 \left(\frac{(n-1)}{2} W_1(\varepsilon) \varepsilon^2 - n W_2(\varepsilon) \mathcal{N}(\varepsilon) \right)$ is non-negative on M_t . In fact, $\mathcal{N}(\varepsilon)$ is a strictly decreasing function of ε ; in addition, $\mathcal{N}(\varepsilon)$ is arbitrarily

large as ε goes to zero and tends to zero as ε goes to $1/n$ by definition, $W_1(\varepsilon)$ is increasing and $W_2(\varepsilon)$ is decreasing. Therefore, $\mathcal{F}(\varepsilon)$ is a strictly increasing function of ε , it is negative as ε goes to zero and positive as ε goes to $1/n$. So there exists a unique value $\varepsilon_0 \in (0, 1/n)$ such that

$$(4.19) \quad \mathcal{F}(\varepsilon_0) = 0.$$

By Lemma 4.2 there exists a constant $C^* \in (0, 1/n^n)$ satisfies $Q_2(\lambda) > C^*$ such that $\lambda_1 > \varepsilon F(\lambda)$ with a $\varepsilon_0 \in (0, 1/n)$ given by (4.19). Thus, if $Q_2 > C^* \geq 0$ everywhere on the initial hypersurface, applying the maximum principle for Q_2 implies that $\partial_t Q \geq 0$, i.e., Q is nondecreasing in time. This guarantees that $Q_2 > C^*$ is preserved on any solution M_t of (1.1)-(1.2) with Φ given by (1.5) in \mathbb{R}^{n+1} . \square

Theorem 4.5 asserts that for convex F inequality $Q_2 > C^*$ ($Q_1 > C^*$ for concave F) holds for all $t \in [0, T)$, furthermore the definition of C^* together with Lemma 4.2 shows that

$$(4.20) \quad \lambda_i \geq \varepsilon_0 n F \geq \varepsilon_0 H \quad (\lambda_i \geq \varepsilon_0 H \geq \varepsilon_0 n F) \quad \text{on } M \times [0, T) \quad \text{for each } i,$$

where ε_0 is given by (4.19). In particular, the fact the solution is convex for all t implies

$$(4.21) \quad \lambda_i \leq H \quad \text{on } M \times [0, T) \quad \text{for each } i.$$

Combining now (4.20) and (4.21), we conclude that M_t satisfies everywhere the pointwise curvature pinching estimate

$$(4.22) \quad \lambda_n \leq C_3 \lambda_1,$$

where $C_3 = \varepsilon_0^{-1}$.

Thus, a uniform double side bound for the inradius and outer radius of the evolving hypersurfaces follows by combining (4.22) with Lemma 2.5 and Lemma 3.3.

Corollary 4.6. *Under the assumptions of Theorem 4.5, there are constants $D_i = D_i(n, \beta, V_{n-m})$, $i \in \{1, 2\}$, such that*

$$D_1 \leq \rho_-(t) \leq \rho_+(t) \leq D_2 \quad \text{for every } t \in [0, T).$$

Also as in [2, Corollary 4.6.] curvature pinching implies that the first derivatives of F has a uniform double side bound.

Corollary 4.7. *Under the assumptions of Theorem 4.5, there are constants $C_i = C_i(n, M_0)$, $i \in \{4, 5\}$, such that*

$$C_4 Id \leq \dot{F}(\mathcal{V}) \leq C_5 Id \quad \text{for every } t \in [0, T).$$

5. Upper bound on $\Phi(F)$

In this section uniform bounds from above on the speed for the flow (1.1)–(1.2) with $\Phi(F)$ given by (1.5) for some $\beta \geq 1$ and for the curvature of the hypersurface are derived, depending only on the initial data. Throughout the section, we assume that the flow satisfies the assumptions of Theorem 4.5. As usual, we denote by $\Omega_t \subset \mathbb{R}^{n+1}$ the region enclosed by M_t . The bounds on curvatures together with the estimates in the next section will imply the long time existence of the flow by well-known arguments. Following the technical procedure in literature, we first show that a ball with fixed center remains inside the evolving Ω_t for a suitable time interval.

Lemma 5.1. *If $B(p_{t_0}, \rho_{t_0}) \subset \Omega_{t_0}$ for some $t_0 \in [0, T]$, where $\rho_{t_0} = \rho_-(t_0)$ is the inradius of M_{t_0} , then there exists some constant $\tau = \tau(n, \beta, V_0) > 0$ such that $B(p_{t_0}, \rho_{t_0}/2) \subset \Omega_t$ for every $t \in [t_0, \min\{t_0 + \tau, T\}]$.*

Proof. Proceeding similarly as in [10, Lemma 8] and [34, Lemma 6.1], Our procedure is to compare the deformation of M_t by the equation (1.1) with a sphere shrinking under a quasilinear flow with speed given by a multiple of the power β of the mean curvature.

For convenience, let $r_B(t)$ be the radius at time t of a sphere $\partial B(p_{t_0}, r_B(t))$ centered at p_{t_0} , evolving according to

$$(5.1) \quad \frac{dr_B(t)}{dt} = -\frac{C_6}{(r_B(t))^\beta}.$$

with the initial condition $r_B(t_0) = \rho_{t_0}$ and a positive constant C_6 to be determined, this ODE has solution

$$r_B(t) = \left(\rho_{t_0}^{\beta+1} - C_6(\beta+1)(t-t_0) \right)^{\frac{1}{\beta+1}}.$$

Thus, the sphere shrinks to $B(p_{t_0}, \rho_{t_0}/2)$ if and only if

$$t - t_0 = \frac{2^{\beta+1} - 1}{C_6(\beta+1)} \left(\frac{\rho_{t_0}}{2} \right)^{\beta+1} =: \tau.$$

Now set $\mathfrak{X}(\cdot, t) = X(\cdot, t) - p_{t_0}$ and $f(x, t) = |\mathfrak{X}(x, t)|^2 - |r_B(t)|^2$, from (1.1) and (5.1), it follows

$$(5.2) \quad \partial_t f = 2(\bar{\phi}(t) - \Phi(F)) \langle \nu_t, \mathfrak{X} \rangle + 2C_6 \left(\frac{1}{r_B} \right)^{\beta-1}.$$

On the other hand, by the Euler's relation we compute

$$\Delta_{\dot{\Phi}} f = F^{\beta-1} \Delta_{\dot{F}} |\mathfrak{X}(x, t)|^2 = -2\Phi \langle \nu_t, \mathfrak{X} \rangle + 2F^{\beta-1} \dot{F}_k^k.$$

Therefore, (5.2) can be rewritten as

$$\partial_t f = \Delta_{\dot{\Phi}} f + 2\bar{\phi}(t) \langle \nu_t, \mathfrak{X} \rangle - 2F^{\beta-1} \dot{F}_k^k + 2C_6 \left(\frac{1}{r_B} \right)^{\beta-1}.$$

As strict convexity holds for each M_t , we have that $\bar{\phi}(t)$, and $\langle \nu_t, \mathfrak{X} \rangle$ are all positive, which allow us to disregard the term containing $\bar{\phi}(t)$. We note that at any point of M_t ,

$$\dot{F}_k^k = \text{tr}(\dot{F}) \leq nC_5,$$

due to Corollary 4.7 and $f(\lambda_1, \dots, \lambda_n) \leq \lambda_n \leq \frac{1}{r_B}$. So if we take $C_6 = nC_5$, then

$$\partial_t f \geq \Delta_{\dot{\Phi}} f.$$

Thus, since $f(x, t_0) \geq 0$ using a standard comparison principle we conclude that

$$(5.3) \quad f(x, t) \geq 0,$$

and hence $B(p_{t_0}, \rho_{t_0}/2) \subset \Omega_t$ for every $t \in [t_0, \min\{t_0 + \tau, T\}]$. which concludes the proof. \square

In order to obtain an upper bound on $\Phi(F)$, as in [42, 33, 34, 10, 12], the method is to study the evolution under the flow of the function

$$Z_t = \frac{\Phi(F)}{\mathcal{S} - \epsilon}.$$

Here $\mathcal{S} = \langle \nu, \mathfrak{X} \rangle$, the position vector field $\mathfrak{X}(\cdot, t) = X(\cdot, t) - p_{t_0}$ (with origin p_{t_0}) on M_t , and ϵ is a constant to be chosen later. Its evolution equation is straightforward to compute using (3.3) and (3.8).

Corollary 5.2. *For $t \in [0, T)$ and any constant ϵ ,*

$$(5.4) \quad \begin{aligned} \partial_t Z = & \Delta_{\dot{\Phi}} Z + \frac{2 \langle \nabla Z, \nabla \mathcal{S} \rangle_{\dot{\Phi}}}{\mathcal{S} - \epsilon} - \epsilon \frac{Z}{\mathcal{S} - \epsilon} \text{tr}_{\dot{\Phi}}(A\mathcal{W}) + (1 + \beta)Z^2 \\ & - \frac{\bar{\phi}}{\mathcal{S} - \epsilon} (\text{tr}_{\dot{\Phi}}(A\mathcal{W})) - \frac{Z}{\mathcal{S} - \epsilon} \bar{\phi}. \end{aligned}$$

The above lemma assists us by allowing us to consider a uniform bound on the speed of the flow.

Corollary 5.3. *For $t \in [0, T)$,*

$$(5.5) \quad \Phi(F)(\cdot, t) < C_7 = C_7(n, \beta, M_0),$$

moreover,

$$(5.6) \quad F(\cdot, t) < C_8 := C_7^{1/\beta}.$$

Proof. For any fixed $t_0 \in [0, T)$, let p_{t_0} and ρ_{t_0} be as in Lemma 5.1. Then, using the convexity of the M_t 's and taking the constant $\epsilon = \rho_{t_0}/4$ leads to

$$(5.7) \quad \mathcal{S} - \epsilon \geq \epsilon > 0,$$

on the same time interval, which ensures $Z_t = \frac{F}{\mathcal{S} - \epsilon}$ is well-defined.

Let us go back to equation (5.4), since strict convexity holds for each M_t , $\Phi(F)$, $\bar{\phi}$ and $\text{tr}_{\dot{\Phi}}(A\mathcal{W})$ are all positive, which together with (5.7), the term containing $\bar{\phi}$ can be neglected. Furthermore, note that $\Phi(F)$ is homogeneous of degree β , Euler's relation and (4.20) together give the following

$$\text{tr}_{\dot{\Phi}(F)}(A\mathcal{W}) = \dot{\Phi}^i \lambda_i^2 \geq \varepsilon_0 n F \dot{\Phi}^i \lambda_i = n \varepsilon_0 \beta F \Phi.$$

Now from the above remark,

$$(5.8) \quad \partial_t Z \leq \Delta_{\dot{\Phi}} Z + \frac{2 \langle \nabla Z, \nabla \Phi \rangle_{\dot{\Phi}}}{\mathcal{S} - \epsilon} - \epsilon \varepsilon_0 \beta \Phi^{\frac{1}{\beta}} Z^2 + (1 + \beta) Z^2.$$

On the other hand, from (5.7) it follows

$$Z \leq \frac{\Phi}{\epsilon}.$$

Applying this to (5.8) gives

$$\partial_t Z \leq \Delta_{\dot{F}} Z + \frac{2 \langle \nabla Z, \nabla \Phi \rangle_{\dot{F}}}{\mathcal{S} - \epsilon} + \left((1 + \beta) - \epsilon^{1 + \frac{1}{\beta}} n \beta \varepsilon_0 Z^{\frac{1}{\beta}} \right) Z^2.$$

Assume that in (\bar{x}, \bar{t}) , $\bar{t} \in [t_0, \min\{t_0 + \tau, T\})$, Z attains a big maximum $C \gg 0$ for the first time. Then

$$Z(\bar{x}, \bar{t}) = C(\mathcal{S} - \epsilon)(\bar{x}, \bar{t}) \geq \epsilon C,$$

which gives a contradiction if

$$C > \max_{x \in M^n} \left\{ Z(x, t_0), \frac{1}{\epsilon} \left(\frac{(\beta + 1)}{n\varepsilon_0\epsilon\beta} \right)^\beta \right\}.$$

Thus,

$$Z_t(x) \leq \max_{x \in M^n} \left\{ Z(x, t_0), \frac{1}{\epsilon} \left(\frac{(\beta + 1)}{n\varepsilon_0\epsilon\beta} \right)^\beta \right\},$$

on $[t_0, \min\{t_0 + \tau, T\})$. Applying the fact that our flow preserves the mixed volume V_{n-m} to Lemma 2.5, we can control the maximal distance between p_{t_0} and the points in $\partial\Omega_t$ by a big positive constant D . Then from the definition of Z_t , it follows

$$\Phi(x, t) \leq (D - \epsilon) \max_{x \in M^n} \left\{ Z(x, t_0), \frac{1}{\epsilon} \left(\frac{c_\kappa(D_2)(\beta + 1)}{n\varepsilon_0\epsilon\beta} \right)^\beta \right\},$$

on $[t_0, \min\{t_0 + \tau, T\})$. Since t_0 is arbitrary, and τ does not depend on t_0 , this implies

$$\begin{aligned} \Phi(x, t) &\leq (D - \epsilon) \max_{x \in M^n} \left\{ Z(x, t_0), \frac{1}{\epsilon} \left(\frac{c_\kappa(D_2)(\beta + 1)}{n\varepsilon_0\epsilon\beta} \right)^\beta \right\} \\ &=: C_7(n, \beta, M_0), \end{aligned}$$

on $[0, T)$, and so (5.6) follows by the definition of $\Phi(F)$. \square

Inserting the estimate of (5.5) into (1.2) immediately gives the following

Corollary 5.4. *For $t \in [0, T)$,*

$$(5.9) \quad \bar{\phi}(t) < C_6.$$

Hence the speed of the evolving hypersurfaces is bounded.

Corollary 5.5. *For $t \in [0, T)$,*

$$\left| \frac{\partial}{\partial t} X(p, t) \right| < C_9 := 2C_6.$$

The curvature of M_t also remains bounded.

Corollary 5.6. *For $t \in [0, T)$,*

$$(5.10) \quad |\mathcal{W}| < H \leq C_{10}.$$

Proof. In the case of concave F , the homogeneity of $\Phi(F)$, Corollary 4.7 and (4.21) imply that

$$\beta\Phi = \dot{\Phi}\lambda_i \geq \varepsilon_0 H \text{tr}(\dot{\Phi}) = \varepsilon_0 H F^{\beta-1} \text{tr}(\dot{F}) \geq \varepsilon_0 H \beta \Phi^{1-\frac{1}{\beta}} c_5.$$

Thus, by (5.5)

$$H \leq \frac{1}{\varepsilon_0 C_4} \Phi^{\frac{1}{\beta}} \leq \frac{1}{\varepsilon_0 c_5} C_6^{\frac{1}{\beta}} =: C_{10},$$

and so with the convexity of M_t

$$|\mathcal{W}| < C_{10}.$$

In the case of concave F , in view of the relation $F \geq \frac{1}{n}H$, we can easily obtain the estimate (5.10). \square

6. Long time existence

In this section, it is shown that solution of the initial value problem (1.1)–(1.2) with $\Phi(F)$ given by (1.5) for some $\beta \geq 1$, with one of the following pinching conditions:

(i). for convex F ,

$$K(p) > \mathcal{C}F^n > 0,$$

(ii). for convex F ,

$$K(p) > \mathcal{C}H^n > 0,$$

exists for all positive times. As usual, the first step is to obtain suitable bounds on the solution on any finite time interval $[0, T)$, which guarantees our problem has a unique solution on the time interval such that the solution converges to a smooth hypersurface M_T as $t \rightarrow T$. Thus, it is necessary to show that the solution remains uniformly convex on the finite time interval which ensure the parabolicity assumption of (1.1).

First it is to show the preserving convexity of the evolving hypersurface M_t . Recall that Theorem 4.5 and Lemma 4.2 together imply the uniform convexity of M_t , however, comparing with the initial assumptions of Theorem 1.3, there is a priori assumption $F > 0$ in Theorem 4.5. As Cabezas-Rivas and Sinestrari mentioned in [12], note that for small times such an assumption holds due to the smoothness of the flow for small times and the initial pinching condition (1.6) or (1.7), but it is possible that at some positive time all $\min K$, $\min F$ and $\min H$ tend to zero such that K/H^n and K/F^n remains bounded. Thus, to exclude such a behaviour, following [12], it is necessary to complement Theorem 4.5 by establishing positivity of F for the finite time.

Lemma 6.1. *Under the hypotheses of Theorem 4.5,*

$$F(\cdot, t) \geq F(\cdot, 0)e^{-C_{11}T} \quad \text{for all times } t \in [0, T).$$

Proof. Let us go back to the evolution equation (3.5) of $\Phi(F)$,

$$\partial_t \Phi = \Delta_{\dot{\Phi}} \Phi + (\Phi - \bar{\phi}) \operatorname{tr}_{\dot{\Phi}}(A\mathcal{W}).$$

Since under the hypotheses of Theorem 4.5, the evolving hypersurfaces M_t remains convex for every $t \in [0, T)$, i.e. $\lambda_i \geq 0$ on $[0, T)$, taking a normal coordinate system at a point where \mathcal{W} is diagonal, we have

$$\operatorname{tr}_{\dot{\Phi}}(A\mathcal{W}) = \sum_{i=1}^n \dot{\Phi}^i \lambda_i^2 \geq 0.$$

Then we have the following computation of Φ :

$$(6.1) \quad \partial_t \Phi \geq \Delta_{\dot{\Phi}} \Phi - \bar{\phi} \sum_{i=1}^n \dot{\Phi}^i \lambda_i^2 \geq \Delta_{\dot{\Phi}} \Phi - C_{11} \sum_{i=1}^n \dot{\Phi}^i \lambda_i,$$

where we have used the estimates (5.5) and (5.9) in the last step, and $C_{11} = C_6(C_{10})$.

Now we use the fact that Φ is homogeneous of degree β , we have the evolving inequality of Φ

$$\partial_t \Phi \geq \Delta_{\dot{\Phi}} \Phi - C_{11}\beta\Phi,$$

The parabolic maximum principle and the fact $\Phi(\cdot, 0) \geq 0$ now give

$$\Phi(\cdot, t) \geq \Phi(\cdot, 0)e^{-C_{11}\beta t}.$$

This also implies that for all times $t \in [0, T)$,

$$F(\cdot, t) \geq F(\cdot, 0)e^{-C_{11}T}.$$

□

Corollary 6.2. *The solution of (1.1)–(1.2) with $\Phi(F)$ given by (1.5) for some $\beta \geq 1$, with one of the following pinching conditions:*

(i). *for convex F ,*

$$K(p) > CF^n > 0,$$

(ii). *for convex F ,*

$$K(p) > CH^n > 0,$$

are uniformly convex on any finite time interval; that is, for any $T < +\infty$, $T \leq T_{\max}$, we have

$$\inf_{M \times [0, T)} \lambda_i > 0, \quad \forall i = 1, \dots, n.$$

Therefore, Theorem 4.5 is valid also without the hypothesis that $F > 0$ for $t \in (0, T)$. The same holds for the other results that have been obtained until here under the same assumptions of Theorem 4.5.

Now to obtain uniform estimates on all orders of curvature derivatives and hence smoothness and convergence of the M_t for the flow, (1.1)–(1.2) with $\Phi(F)$ given by (1.5) we use a more PDE theoretic approach, following an argument similar to the one in [12].

Since the uniformly bound the speed of our flow we have just obtained implies a bound on all principal curvatures, as in [12] (see also [37, Lemma 3.4]), in view of Lemma 5.1, one can adopt a local graph representation for a convex hypersurface with uniformly bounded C^2 norm.

Corollary 6.3. *There exists r (depending only on $\max F$) with the following property. Given any $(p_{t_0}, t_0) \in M \times (0, T)$, there is a neighborhood \mathcal{U} of the point $x_0 := X(p_{t_0}, t_0)$ such that $M_t \cap \mathcal{U}$ coincides with the graph of a smooth function*

$$u : B_r \times J \rightarrow \mathbb{R}, \quad \text{for all } t \in J.$$

Here $B_r \subset T_{p_{t_0}} M_{t_0}$ is the ball of radius r centered at x_0 in the hyperplane tangent to M_{t_0} at x_0 , and J is the time interval $J = [t_0, \min\{t_0 + \tau, T\})$ (near to t_0), $p_{t_0} \in \Omega_t$. In addition, the C^2 norm of u is uniformly bounded (by a constant depending only on $\max F$).

In graphical coordinates, we see that

$$(6.2) \quad X(p, t) = (x(p, t), u(x(p, t), t)).$$

So the metric and its inverse (cf. [12]) are given by

$$g_{ij} = \delta_{ij} + D_i u D_j u, \quad g^{ij} = \delta^{ij} - \frac{D^i u D^j u}{1 + |Du|^2},$$

where D_i denote the derivatives with respect to these local coordinates, respectively. The outward unit normal vector of M_t can be expressed as

$$\nu = \frac{1}{|\xi|} \begin{pmatrix} -Du, 1 \end{pmatrix}$$

with

$$|\xi| = \sqrt{1 + |Du|^2}.$$

$$h_{ij} = \frac{D_{ij}u}{(1 + |Du|^2)^{1/2}},$$

and

$$h_j^i = \left(\delta^{ik} - \frac{D^i u D^k u}{1 + |Du|^2} \right) \frac{D_{kj}u}{(1 + |Du|^2)^{1/2}},$$

In addition, the Christoffel symbols have the expression:

$$\Gamma_{ij}^k = \left(\delta^{kl} - \frac{D^k u D^l u}{1 + |Du|^2} \right) D_{ij}u D_l u.$$

Theorem 6.4. *Under the hypotheses of Theorem 1.3, for any $0 < t_0 < T$, $\alpha \in (0, 1)$ and every natural number k , there exist constant C_{14} , depending on (n, β, M_0, k) such that*

$$(6.3) \quad \|u\|_{C^k(M \times (t_0, T])} \leq C_{14}.$$

Proof. Due to a tangential diffeomorphism into the flow (1.1), u then satisfies the following parabolic PDE

$$(6.4) \quad \partial_t u = |\xi|(\Phi_t - \bar{\phi}_t).$$

It is easy to see that along the flow (1.1), in the coordinate system the corresponding evolution equations for u and Φ , at $(x, t) \in B_r \times J$, can be expressed as

$$(6.5) \quad \partial_t u = F^{\beta-1} g^{ik} \dot{F}_{ij}(\mathcal{W}) D_k D_j u - |\xi| \bar{\phi}_t$$

and

$$(6.6) \quad \begin{aligned} \partial_t \Phi = F^{\beta-1} g^{ik} \dot{F}_{ij}(\mathcal{W}) D_k D_j \Phi - F^{\beta-1} g^{ik} \dot{F}_{ij}(\mathcal{W}) \Gamma_{kj}^l D_{it} \Phi \\ + (\Phi_t - \bar{\phi}_t) F^{\beta-1} g^{ik} g^{lm} \dot{F}_{ij}(\mathcal{W}) h_{il} h_{jm}, \end{aligned}$$

respectively. Since we have uniform bounds on the curvatures both from above and from below on any finite time interval, in view of Corollary 4.6, Corollary 4.7, equations (5.5) and (5.9), equations (6.5) and (6.6) are uniformly parabolic with uniformly bounded coefficients. Then applying Theorem 2.8 to (6.6) gives that for any $0 < r' < 1$ and $J' \subset J$ there exist some constants $C_{12} > 0$ and $\alpha \in (0, 1)$ depending on n, m, β, M_0 such that

$$(6.7) \quad \|\Phi\|_{C^\alpha(B_{r'} \times J')} \leq C_{12},$$

$$(6.8) \quad \|u\|_{C^\alpha(B_{r'} \times J')} \leq C_{12}.$$

Now for any fixed time $t_1 \in J'$, we want to follow an argumentation similar to the one used in [12, Lemma 6.3] (also as in [34]). Then in the case of concave F , the Bellman's extension \bar{F} of F is given by

$$\bar{F}(\bar{\lambda}) := \inf_{\lambda \in \hat{\Gamma}} [F(\lambda) + DF(\lambda)(\bar{\lambda} - \lambda)],$$

for any $\bar{\lambda} \in \Gamma_+$, while in the case of convex F , \bar{F} takes the form

$$\bar{F}(\bar{\lambda}) := \sup_{\lambda \in \hat{\Gamma}} [F(\lambda) + DF(\lambda)(\bar{\lambda} - \lambda)].$$

Notice that F is homogeneous of degree one, the extension simplifies to

$$\bar{F}(\bar{\lambda}) = \begin{cases} \inf_{\lambda \in \hat{\Gamma}} DF(\lambda) \bar{\lambda} & \text{for } F \text{ concave,} \\ \sup_{\lambda \in \hat{\Gamma}} DF(\lambda) \bar{\lambda} & \text{for } F \text{ convex.} \end{cases}$$

Since the Bellman extension is the infimum or the supremum of linear functions, it preserves concavity or convexity, by definition and homogeneity. Importantly, \bar{F} coincides with F on $\hat{\Gamma}$ by homogeneity of F . Furthermore, using the definition of \bar{F} and Corollary 4.7, \bar{F} is uniformly elliptic, that is

$$C_5|\bar{\eta}| \leq \bar{F}(\bar{\lambda} + \bar{\eta}) - \bar{F}(\bar{\lambda}) \leq \sqrt{n}C_5|\bar{\eta}|, \quad \text{for all } \bar{\lambda}, \bar{\eta} \in \mathbb{R}^n, \bar{\eta} \geq 0.$$

Let $V_{t_1} : B_{r'} \rightarrow \mathbb{R}$ be given by

$$V_{t_1}(x) = \frac{\partial_t u(x, t_1)}{\sqrt{1 + |Du(x, t_1)|^2}}.$$

In view of equations (6.4) and (6.7), we also have

$$\|V_{t_1}\|_{C^\alpha(B_{r'})} \leq C_{12}.$$

Thus, we have the corresponding elliptic equation for u

$$\bar{F}^\beta(D^2u(x, t_1), u(x, t_1)) = V_{t_1}(x)$$

where

$$V_{t_1}(x) = (\Phi(x, t_1) - \bar{\phi}(t_1)),$$

satisfies the conditions of Theorem 3 from [8]. Thus, this theorem gives that $\|u\|_{C^{2,\alpha}(B_{r'})} \leq C_{13}$. Therefore, covering M_t by finite many graphs on balls of radius r' implies that

$$\|u\|_{C^{2,\alpha}(M)} \leq C_{13}.$$

From this estimate on $u(\cdot, t)$ for any fixed t_1 , following the procedure of [41, Theorem 2.4] to equation (6.4), a $C^{2,\alpha}$ estimate for u with respect to both space and time can be obtained. Once $C^{2,\alpha}$ regularity is established, combining standard parabolic theory with our short time existence result yields uniform C^k estimates (6.3) for any integer $k > 2$, which implies uniform C^k estimates (6.3) for any integer $k \geq 1$ with the fact that u and its first order derivatives are uniformly bounded. \square

Theorem 6.5. *Under the hypotheses of Theorem 1.3, the solution M_t of (1.1) with initial condition X_0 , exists, is smooth and satisfies at every point the initial pinching condition on $[0, \infty)$.*

Proof. As we know, the preserving pinching condition and smoothness throughout the interval of existence follows from Theorem 4.5, and Lemma 6.1.

On the other hand, it is clear that from the expression (6.2) that all the higher order derivatives of X_t are bounded if and only if the corresponding derivatives of u are bounded. Thus, uniform C^k estimate (6.3) of u implies that all the derivatives of X_t are also uniformly bounded.

It remains to show that the interval of existence is infinite. Suppose to the contrary there is a maximal finite time T beyond which the solution cannot be extended. Then the evolution equation (1.1) implies that

$$\|X(p, \sigma) - X(p, \tau)\|_{C^0(X_0)} \leq \int_\tau^\sigma |\bar{\phi} - \Phi|(p, t) dt$$

for $0 \leq \tau \leq \sigma < T$. By (5.5) and (5.9), $X(\cdot, t)$ tends to a unique continuous limit $X(\cdot, T)$ as $t \rightarrow T$. In order to conclude that $X(\cdot, T)$ represents a hypersurface M_T ,

next under this assumption and in view of the evolution equation (3.1) the induced metric g remains comparable to a fix smooth metric \tilde{g} on M^n :

$$\left| \frac{\partial}{\partial t} \left(\frac{g(u, u)}{\tilde{g}(u, u)} \right) \right| = \left| \frac{\partial_t g(u, u)}{g(u, u)} \frac{g(u, u)}{\tilde{g}(u, u)} \right| \leq 2|\Phi(x, t_1) - \bar{\phi}| |A|_g \frac{g(u, u)}{\tilde{g}(u, u)},$$

for any non-zero vector $u \in TM^n$, so that ratio of lengths is controlled above and below by exponential functions of time, and hence since the time interval is bounded, there exists a positive constant C such that

$$\frac{1}{C} \tilde{g} \leq g \leq C \tilde{g}.$$

Then the metrics $g(t)$ for all different times are equivalent, and they converge as $t \rightarrow T$ uniformly to a positive definite metric tensor $g(T)$ which is continuous and also equivalent by following Hamilton's ideas in [24]. Therefore using the smoothness of the hypersurfaces M_t and interpolation,

$$\begin{aligned} & \|X(p, \sigma) - X(p, \tau)\|_{C^k(X_0)} \\ & \leq C \|X(p, \sigma) - X(p, \tau)\|_{C^0(X_0)}^{1/2} \|X(p, \sigma) - X(p, \tau)\|_{C^{2k}(X_0)}^{1/2} \\ & \leq C |\sigma - \tau|^{1/2}, \end{aligned}$$

so the sequence $\{X(t)\}$ is a Cauchy sequence in $C^k(X_0)$ for any k . Therefore M_t converges to a smooth limit hypersurface M_T which must be a compact embedded hypersurface in \mathbb{R}^{n+1} . Finally, applying the local existence result, the solution M_T can be extended for a short time beyond T , since there is a solution with initial condition M_T , contradicting the maximality of T . This completes the proof of Theorem 6.5. \square

7. Exponential convergence to the sphere

Observe that, to finish the proof of Theorem 1.3, it remains to deal with the issues related to the convergence of the flow (1.1): It is proved that solutions of equation (1.1) with initial conditions converge, exponentially in the C^∞ -topology, to a sphere in \mathbb{R}^{n+1} as t approaches infinity, whether F be convex or concave.

Since our previous results depend on the lower bound for F in time T , we know that the solution is smooth and uniformly convex on any finite time interval, but we cannot completely control its behavior as $t \rightarrow \infty$. So at this stage we cannot exclude that $\min_{M_t} F \rightarrow 0$ as $t \rightarrow \infty$ so that uniform convexity is lost. Since the regularity estimates depend on uniform convexity, some additional argument is needed to ensure the existence of a smooth limit as $t \rightarrow \infty$.

We first want to show that, if a smooth limit exists, the solution of (1.1) has to be a round sphere; the existence of the limit will be proved afterwards. To address the first step, as in Theorem 4.5 we consider the quotient Q_2 , that is K/F^n , for convex F and the quotient Q_1 , that is K/H^n , for concave F .

To understand the behaviour of the quotients Q_i , $i = 1, 2$, from their evolution equations (4.1) and (4.2) by application of the maximum principle, previously we need to check that the global term $\bar{\phi}(t)$ has the right sign.

Lemma 7.1. *There exists a constant $\bar{\phi}_0 = \bar{\phi}_0(n, m, \beta,) > 0$ such that $\bar{\phi}(t) \geq \bar{\phi}_0$ for all $t \geq 0$, whether F be convex or concave.*

Proof. In view of estimates (4.20), we can use Lemma 2.6 *iii*) to estimate

$$\begin{aligned}
\left(\int_M E_{m+1} d\mu_t \right)^{\frac{\beta+m+1}{m+1}} &\leq |M_t|^{\frac{\beta}{m+1}} \int_M (E_{m+1})^{\frac{\beta+m+1}{m+1}} d\mu_t \\
&\leq |M_t|^{\frac{\beta}{m+1}} \int_M (nH)^\beta E_{m+1} d\mu_t \\
&\stackrel{(4.20)}{\leq} \varepsilon_0^{-\beta} |M_t|^{\frac{\beta}{m+1}} \int_M (\lambda_1)^\beta E_{m+1} d\mu_t \\
&\leq \varepsilon_0^{-\beta} |M_t|^{\frac{\beta}{m+1}} \int_M \Phi(F) E_{m+1} d\mu_t
\end{aligned}$$

where we have applied a Hölder inequality in the first inequality. Hence

$$\begin{aligned}
\bar{\phi}(t) &= \frac{\int_M \Phi(F) E_{m+1} d\mu_t}{\int_M E_{m+1} d\mu_t} \\
(7.1) \quad &\geq \varepsilon_0^\beta \left(\frac{\int_M E_{m+1} d\mu_t}{|M_t|} \right)^{\frac{\beta}{m+1}} = \left(\frac{V_{n-m-1}}{V_n} \right)^{\frac{\beta}{m+1}}.
\end{aligned}$$

Now, the fact the mixed volume V_{n-m} is fixed along the flow and Corollary 4.6 implies that the area V_n of M_t is not greater than the area of a sphere of radius c_2 , while the mixed volume V_{n-m-1} of M_t is greater than the V_{n-m-1} of a sphere of radius D_1 . The use of this in the expression on the right hand side of (7.1) gives the desired lower bound for $\bar{\phi}$. \square

Now we define $F_{\min}(t) = \min_M F(\cdot, t)$. In the next lemma we will show that F_{\min} does not decay too fast, and this will be enough for our purposes.

Proposition 7.2. *We have*

$$\int_0^\infty F_{\min}(t) dt = +\infty,$$

whether F be convex or concave.

Proof. Let us first estimate $\mathcal{P}(t) := \min_M \Phi(\cdot, t)$. Since our evolving manifolds are convex, equation (6.1) gives

$$\begin{aligned}
(\partial_t - \Delta_\Phi) \Phi &\geq -\bar{\phi} \sum_{i=1}^n \dot{\Phi}^i \lambda_i^2 \stackrel{(5.9)}{\geq} -C_7 \sum_{i=1}^n \dot{\Phi}^i \lambda_i^2 \\
&\stackrel{(4.20)}{\geq} -C_7 \lambda_n \sum_{i=1}^n \dot{\Phi}^i \lambda_i \stackrel{(4.22)}{\geq} -C_7 C_3 F \sum_{i=1}^n \dot{\Phi}^i \lambda_i \\
&= -C_7 C_3 \beta \Phi^{1+\frac{1}{\beta}} =: -\tilde{C} \Phi^{1+\frac{1}{\beta}},
\end{aligned}$$

where we have used homogeneity of Φ and normalisation of F . Therefore by applying the maximum principle to the above inequality we have

$$\mathcal{P} \geq \left(\mathcal{P}(0)^{-\frac{1}{\beta}} + \tilde{C} \frac{t}{\beta} \right)^{-\beta},$$

from which it follows that

$$\int_0^\infty F_{\min}(t) dt = \lim_{s \rightarrow \infty} \int_0^s n \mathcal{P}^{\frac{1}{\beta}}(t) dt \geq \lim_{s \rightarrow \infty} \int_0^s n \left(\mathcal{P}(0)^{-\frac{1}{\beta}} + \tilde{C} \frac{t}{\beta} \right)^{-1} dt = \infty.$$

□

Remark 7.3. Since $H \geq nF$ in the case of concave F , we have

$$\int_0^\infty H_{\min}(t)dt = +\infty.$$

The following result shows that if the limiting hypersurfaces exists, it has to be a round sphere.

Proposition 7.4. *The shape of M_t approaches the shape of a round sphere as $t \rightarrow \infty$.*

Proof. In the case of convex F , in the proof of Theorem 4.5, by applying the weak maximum principle to (4.10) for $\mathcal{Q}(t) = \min_{M_t} \left(\frac{K}{F^n} \right)$, we have $\mathcal{Q}(t) \geq \mathcal{Q}(0)$. Now to analysis the shape of M_t as $t \rightarrow \infty$, we apply the maximum principle to the evolution equation (4.2). Suppose Q_2 attained a new minimum at some (p_0, t_0) , $t_0 > 0$. the strong maximum principle then implies that Q_2 is identically constant. If this is the case, from equation (4.2) we must have

$$\begin{aligned} 0 &\equiv \frac{1}{F^2} |F\nabla\mathcal{W} - \mathcal{W}\nabla F|_{\Phi, b}^2 + \text{tr}_{b - \frac{n}{F}\dot{F}} \left(\ddot{\Phi}(\nabla\mathcal{W}, \nabla\mathcal{W}) \right) \\ &\quad + n\frac{1}{F}\ddot{F}(\nabla\mathcal{W}, \nabla\mathcal{W}) + [(\beta - 1)\Phi + \bar{\phi}] \frac{1}{F} (n\text{tr}_{\dot{F}}(A\mathcal{W}) - HF). \end{aligned}$$

Since each of the sum of the first two terms, the third term and the last term are nonnegative, each must be identically equal to zero. Thus, from the fact that $\bar{\phi}$ has a positive lower bound, M_t is convex and $\beta \geq 1$, the last expression in brackets must equal to zero, this means that at any point of M_t ,

$$\begin{aligned} 0 &\equiv n\text{tr}_{\dot{F}}(A\mathcal{W}) - HF \\ &= \sum_{ij} \left(\dot{f}_i \lambda_i^2 - \lambda_i \dot{f}_j \lambda_j \right) = \sum_{i \neq j} \left(\dot{f}_i \lambda_i - \dot{f}_j \lambda_j \right) (\lambda_i - \lambda_j). \end{aligned}$$

Now for any pair $i \neq j$ with $\lambda_i \neq \lambda_j$, Lemma 2.2 implies

$$\left(\dot{f}_i \lambda_i - \dot{f}_j \lambda_j \right) (\lambda_i - \lambda_j) \neq 0.$$

Thus,

$$\sum_{i \neq j} \left(\dot{f}_i \lambda_i - \dot{f}_j \lambda_j \right) (\lambda_i - \lambda_j) \neq 0.$$

So we must have $\lambda_i = \lambda_j$ for all i and j . Thus, M_{t_0} is umbilical everywhere, and therefore is a round sphere.

In the case of concave F , a similar strong maximum principle argument on the evolution equation (4.1) as above again shows that the shape of M_t as $t \rightarrow \infty$ is a round sphere. □

Since the speed for the flow has a homogeneity degree larger than one in the curvatures, in contrast with the standard Laplacian Δ , the operators $\Delta_{\Phi} = \beta F^{\beta-1} \Delta_{\dot{F}}$ which appear in the evolution equations become degenerate if the curvatures approach zero as $t \rightarrow \infty$. To exclude such a possibility, unlike the previous approach for various geometric flow with higher homogeneity speeds by using an interior estimate on porous medium equations, we use an argument similar to Andrews and McCoy [5] to obtain a positive lower speed bounds on finite time intervals, after a sufficient big time.

Proposition 7.5. Set $\varpi_{t_0}(t) = \int_{t_0}^t \bar{\phi}(s) ds$. Under the flow (1.1), for any t_0 , there exists a positive constant γ such that

$$\Phi \geq \frac{\rho_- + \varpi_{t_0} - R_{t_0}(t)}{\gamma(t - t_0)},$$

where $R_{t_0}(t)$ is the radius of a ball that encloses M_{t_0} and evolves under the flow (1.1) with the same $\bar{\phi}$ as that of the evolving hypersurface M_t . In particular, Choosing t_0 big enough, we have a positive lower bound C_{16} for Φ after a given waiting time.

Proof. We begin with the following observation, which originate from the work of Smoczyk [40, Proposition 4] for the mean curvature flow.

Lemma 7.6. If M_{t_0} enclosed $p \in \mathbb{R}$, then under the flow (1.1) for any t_0 there exists a positive constant γ such that for a short time later

$$\langle X - p, \nu \rangle + \gamma(t - t_0)\Phi - \varpi_{t_0} > 0.$$

Proof of the lemma. From (3.3) and (3.8), we have the evolution equation

$$\begin{aligned} (7.2) \quad & \partial_t [\langle X - p, \nu \rangle + \gamma(t - t_0)\Phi - \varpi_{t_0}] \\ &= \Delta_{\dot{\Phi}} [\langle X - p, \nu \rangle + \gamma(t - t_0)\Phi - \varpi_{t_0}] \\ &+ [\langle X - p, \nu \rangle + \gamma(t - t_0)\Phi - \varpi_{t_0}] \operatorname{tr}_{\dot{\Phi}}(A\mathcal{W}) \\ &+ [\varpi_{t_0} - \gamma(t - t_0)\bar{\phi}] \operatorname{tr}_{\dot{\Phi}}(A\mathcal{W}) + [\gamma - (\beta + 1)]\Phi. \end{aligned}$$

Since

$$\operatorname{tr}_{\dot{\Phi}}(A\mathcal{W}) = \sum_{i=1}^n \dot{\Phi}^i \lambda_i^2 \leq \lambda_n \sum_{i=1}^n \dot{\Phi}^i \lambda_i \stackrel{(4.22)}{\leq} C_3 F \sum_{i=1}^n \dot{\Phi}^i \lambda_i = C_3 \beta \Phi^{1+\frac{1}{\beta}},$$

we estimate

$$\begin{aligned} & [\varpi_{t_0} - \gamma(t - t_0)\bar{\phi}] \operatorname{tr}_{\dot{\Phi}}(A\mathcal{W}) + [\gamma - (\beta + 1)]\Phi \\ & \geq -\gamma(t - t_0)\bar{\phi} \operatorname{tr}_{\dot{\Phi}}(A\mathcal{W}) + [\gamma - (\beta + 1)]\Phi \\ & \geq -\gamma(t - t_0)\bar{\phi} C_3 \beta \Phi^{1+\frac{1}{\beta}} + [\gamma - (\beta + 1)]\Phi \\ & \stackrel{(5.6), (5.9)}{\geq} -\gamma(t - t_0) C_7^{1+\frac{1}{\beta}} C_3 \beta \Phi + [\gamma - (\beta + 1)]\Phi \\ & = \left[\left[1 - (t - t_0) C_7^{1+\frac{1}{\beta}} C_3 \beta \right] \gamma - (\beta + 1) \right] \Phi. \end{aligned}$$

So taking $C_{15} = C_7^{1+\frac{1}{\beta}} C_3 \beta$ and $\gamma = 2(\beta + 1)$, we can deduce that this is positive for $t \in [t_0, t_0 + \frac{1}{2C_{15}}]$. Thus, applying the maximum principle to (7.2) gives the result of the lemma. \square

Proof of the proposition, continued. For any given positive constant ε , choose t_1 big enough to ensure $\lambda_n(M_t) \leq (1 + \varepsilon)\lambda_1(M_t)$ for all $t \in [t_1, \infty]$. Fix $t_0 \in [t_0, \infty]$. Choose $q \in \mathbb{R}$ to be the incentre of M_{t_0} such that M_{t_0} encloses B_{ρ_-} . Let $R_{t_0}(t_0) = \rho_+$, which denotes the outer radius of M_{t_0} , it follows that $B_{R_{t_0}(t_0)}$ encloses M_{t_0} . For a given z , choose p to be the point in M_{t_0} which maximizes $\langle p, \nu(z, t) \rangle$, then we have $\langle p, \nu(z, t) \rangle \geq \rho_-$ for $t > t_0$. Let $R_{t_0}^+(t) = \rho_- + \varpi_{t_0}(t)$. Now we consider

the two evolving spheres, whose radii satisfy

$$(7.3) \quad \begin{cases} \frac{\partial}{\partial t} R_{t_0}^+ = \bar{\phi}(t), \\ R_{t_0}^+(t_0) = \rho_-, \end{cases}$$

and

$$(7.4) \quad \begin{cases} \frac{\partial}{\partial t} R_{t_0} = \bar{\phi}(t) - R_{t_0}^{-\beta}(t), \\ R_{t_0}(t_0) = \rho_+, \end{cases}$$

respectively. Thus from (7.3) and (7.4), we get

$$(7.5) \quad \begin{cases} \frac{\partial}{\partial t} (R_{t_0}^+ - R_{t_0}) = R_{t_0}^{-\beta}(t), \\ R_{t_0}^+(t_0) - R_{t_0}(t_0) = \rho_- - \rho_+, \end{cases}$$

Since R_{t_0} is greater than the radius of the sphere with the same value of V_{n-m} as the evolving hypersurface M_t from (7.4), so $\bar{\phi}(t) > R_{t_0}^{-\beta}(t)$. We deduce that R_{t_0} is increasing on the time interval $[t_0, t_0 + \frac{1}{2C_{15}}]$. In particular, combining this with (5.9), we further have $R_{t_0} \leq \rho_+ + \frac{C_7}{2C_{15}}$ by (5.9). From (7.5) this implies that on $[t_0, t_0 + \frac{1}{2C_{15}}]$, we have

$$R_{t_0}^+(t) - R_{t_0}(t) \geq \rho_- - \rho_+ + \left(\rho_+ + \frac{C_7}{2C_{15}} \right)^{-\beta} (t - t_0),$$

which means that

$$\rho_- + \varpi_{t_0} - R_{t_0}(t) \geq \rho_- - \rho_+ + \left(\rho_+ + \frac{C_7}{2C_{15}} \right)^{-\beta} (t - t_0).$$

So if $(t - t_0) > \left(\rho_+ + \frac{C_7}{2C_{15}} \right)^{\beta} (\rho_+ - \rho_-)$, the quantity $\rho_- + \varpi_{t_0} - R_{t_0}(t)$ is positive. In fact, using Proposition 7.4 we may proceed exactly as in [5, Theorem 3.1] to show that $(t - t_0) > \left(\rho_+ + \frac{C_7}{2C_{15}} \right)^{\beta} (\rho_+ - \rho_-)$ can be ensured by the pinching condition $\lambda_n(M_t) \leq (1+\varepsilon)\lambda_1(M_t)$ for any small ε . Now we apply the above lemma to estimate that for $t \in [t_0, t_0 + \frac{1}{2C_{15}}]$ and $\gamma = 2(\beta + 1)$

$$\begin{aligned} \Phi &\geq \frac{\langle p - X(z, t), \nu(z, t) \rangle + \varpi_{t_0}(t)}{\gamma(t - t_0)} \\ &\geq \frac{\rho_- + \varpi_{t_0} - R_{t_0}(t)}{\gamma(t - t_0)} \\ &\geq \frac{\rho_- - \rho_+ + \left(\rho_+ + \frac{C_7}{2C_{15}} \right)^{-\beta} (t - t_0)}{\gamma(t - t_0)} \end{aligned}$$

Thus, our choice of any small ε as required that t_1 is big enough can ensure that

$$\frac{1}{2C_{15}} > \left(\rho_+ + \frac{C_7}{2C_{15}} \right)^{-\beta} (t - t_0).$$

This imply a positive lower bound for Φ after a given waiting time, depending n , β and M_0 . \square

Remark 7.7. We can follow an idea of [38, 12] to show that the evolution equation for F , which in a local coordinate system along (1.1) can be written as

$$\partial_t F = D_i (F^{ij} D_j F^\beta) - \partial_j F^{ji} D_i F^\beta + \Gamma_{jl}^j F^{li} D_i F^\beta + (F^\beta - \bar{\phi}) \text{tr}_{\dot{F}}(A\mathcal{W}),$$

where D_i denote derivatives with respect to the coordinates. This can be view as a porous medium equation as in [38, 12]. Unfortunately, we can not bound the coefficient Γ_{jl}^j in this form and therefore an interior Hölder estimate for solutions of such equations established by DiBenedetto and Friedman ([15, Theorem 1.2]) can not applied to prove a uniform C^α -estimate for F .

The next step is to show that the convergence to a sphere of \mathbb{R}^{n+1} as $t \rightarrow \infty$ is exponential. To address this, we consider the following function

$$f = \frac{1}{n^n} - \frac{K}{H^n}.$$

Then as remarked in Section 4, $f \geq 0$ with equality holding only at umbilic points, which is the value assumed on a sphere. The following Lemma is an immediate consequence of the evolution equation (4.2) of Q_1 .

Lemma 7.8. *The quantity f evolves under (1.1) satisfies*

$$\begin{aligned} \partial_t f = & \Delta_{\dot{\Phi}} f + \frac{(n+1)}{nH^n} \langle \nabla f, \nabla H^n \rangle_{\dot{\Phi}} - \frac{(n-1)}{nK} \langle \nabla f, \nabla K \rangle_{\dot{\Phi}} - \frac{H^n}{nK} |\nabla f|_{\dot{\Phi}}^2 \\ (7.6) \quad & - \left(\frac{Q_1}{H^2} |H \nabla \mathcal{W} - \mathcal{W} \nabla H|_{\dot{\Phi}, b}^2 + Q_1 \text{tr}_{b - \frac{n}{H} Id} \left(\ddot{\Phi}(\nabla \mathcal{W}, \nabla \mathcal{W}) \right) \right) \\ & - [(\beta - 1)\Phi + \bar{\phi}] \frac{Q_1}{H} (n|A|^2 - H^2). \end{aligned}$$

Corollary 7.9. *Under the conditions of Theorem 1.3,*

$$\begin{aligned} (7.7) \quad \partial_t f \leq & \Delta_{\dot{\Phi}} f + \frac{(n+1)}{nH^n} \langle \nabla f, \nabla H^n \rangle_{\dot{\Phi}} - \frac{(n-1)}{nK} \langle \nabla f, \nabla K \rangle_{\dot{\Phi}} - \frac{H^n}{nK} |\nabla f|_{\dot{\Phi}}^2 \\ & - C_{16} H f, \end{aligned}$$

where $C_{16} = \delta C^*$

Proof. For concave F , applying the similar argument as in Theorem 4.5, Lemma 7.1, inequality (1.7) and Lemma 2.7 to (7.6) gives the conclusion. For convex F , we only instead use the curvature pinching inequality (1.6) and note that $\frac{K}{F^n} \leq \frac{n^n K}{H^n}$. \square

Now we can show exponential convergence to the sphere.

Proposition 7.10. *Under the conditions of Theorem 1.3, the rate of convergence of f to 0 as $t \rightarrow \infty$ is exponential.*

Proof. For concave F , applying the maximal principal to (7.7), by using the fact $F \leq \frac{H}{n}$ and Proposition 7.5 gives

$$\partial_t f_{\max}(t) \leq -C_{17} f_{\max}(t).$$

which implies that

$$f_{\max}(t) \leq C_{18} e^{-C_{17} t},$$

where $C_{18} = f_{\max}(0)$. For convex F , we only instead use that $H \geq n\varepsilon_0 F$ and Proposition 7.5. This proves the Proposition. \square

Now if t_0 can be taken big enough so that M_t can be represented as graph u for $[t_0, \infty)$, then since all the derivatives of u are uniformly bounded independent of time, applying Arzel-Ascoli Theorem we conclude that the graph $u(t, \cdot)$ is defined on $[t_0, \infty)$ and converges to a unique function u_∞ . By standard argument as in [11, page 467] this implies that X_∞ is an immersion, and since the convergence is smooth we can assure that X_∞ must be a compact embedded hypersurface. On the other hand, Proposition 7.4 says that all points on X_∞ are umbilic points. In conclusion, the only possibility is that \mathcal{S} represents a sphere in \mathbb{R}^{n+1} and, by the volume-preserving properties of the flow, such a sphere has to enclose the same volume as the initial condition $X_0(M)$.

Finally, from Proposition 7.10 we can conclude with the standard arguments as in [38, Theorem 3.5], [12, Theorem 7.3] that the flow converges exponentially to the sphere \mathcal{S} in \mathbb{R}^{n+1} in the C^∞ -topology.

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SCHOOL OF MATHEMATICS AND STATISTICS, MINNAN NORMAL UNIVERSITY, ZHANGZHOU, 363000, PEOPLE'S REPUBLIC OF CHINA, AND SCHOOL OF MATHEMATICS, SICHUAN UNIVERSITY, CHENGDU 610065, PEOPLE'S REPUBLIC OF CHINA

E-mail address: `guoshunzi@yeah.net`